RESEARCH PROPOSAL: BOREL EQUIVALENCE RELATIONS

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The study of definable equivalence relations on complete separable metric spaces (i.e., *Polish spaces*) has emerged as a new direction of research in descriptive set theory over the past twenty years. Quotients of such equivalence relations include the orbit space of an ergodic group action, the set of Turing degrees of subsets of natural numbers, and the space of isomorphism classes of countable models of some $\mathcal{L}_{\omega_1,\omega}$-sentence, among many others. For all but the simplest equivalence relations E on a Polish space X, the collection X/E of E-classes cannot be viewed in any reasonable way as a definable set inside a Polish space, and so the standard techniques of descriptive set theory do not apply in the usual way to their study. The field of *Borel equivalence relations* seeks to understand the structure of these "singular" spaces X/E. Its methods, as they have been developed over the past decade, draw upon diverse areas of mathematics such as model theory, topology, ergodic theory and orbit equivalence theory, as well as the theory of various classes of countable groups and their definable actions, including especially the free groups, amenable groups, Kazhdan groups, Polish groups, and Lie groups together with their lattice subgroups.

One reason for studying the moduli spaces X/E is that they appear as sets of invariants for classification problems arising throughout mathematics. Indeed, a wide range of naturally occurring classes of mathematical objects may be given the structure of a *standard Borel space* — ie, a Polish space equipped only with its σ-algebra of Borel sets — and in many cases the corresponding notion of classification turn out to be a definable equivalence relation on that space. Consider, for instance, the problem of classifying countable graphs up to isomorphism. Letting $\mathcal{C}$ be the set of graphs of the form $\Gamma = \langle \mathbb{N}, E\rangle$ and identifying each graph $\Gamma \in \mathcal{C}$ with its edge relation $E \in 2^{\mathbb{N}^2}$, one may check that $\mathcal{C}$ is a Borel subset of the Polish space $2^{\mathbb{N}^2}$, and hence is itself a standard Borel space. Furthermore, the isomorphism relation on $\mathcal{C}$ is simply the orbit equivalence relation arising from the natural action of the infinite symmetric group $\mathrm{Sym}(\mathbb{N})$ on $\mathcal{C}$. More generally, if σ is any $\mathcal{L}_{\omega_1,\omega}$-sentence of a countable language $\mathcal{L}$, then

$$\mathrm{Mod}(\sigma) = \{\mathcal{M} \mid \text{the universe of } \mathcal{M} \text{ is } \mathbb{N} \text{ and } \mathcal{M} \models \sigma\}$$

is a standard Borel space, and the isomorphism relation $\cong_\sigma$ on $\mathrm{Mod}(\sigma)$ is the orbit equivalence relation generated by the $\mathrm{Sym}(\mathbb{N})$-action. It can be shown that $\cong_\sigma$ is analytic as a subset of $\mathrm{Mod}(\sigma) \times \mathrm{Mod}(\sigma)$, and that if the collection of objects defined by σ is "finitely generated" in some broad sense then it will in fact be Borel. (For instance, the graph isomorphism relation restricted to countable, connected, *locally finite* graphs is Borel). With these examples in mind we make the following definitions.

Definition 1. *An equivalence relation $E \subseteq X \times X$ on a standard Borel space X is called* Borel *(*analytic*, etc.) if it is Borel (analytic, etc) as a subset of $X \times X$. If E and F are Borel equivalence relations on the standard Borel spaces X, Y respectively, then we call a Borel function $f : X \to Y$ a* Borel reduction *from E to F if for all $x, y \in X$,*

$$x \, E \, y \iff f(x) \, F \, f(y).$$

If such a function exists then we write

$$E \leq_B F$$

and say that E is Borel reducible *to F. If $E \leq_B F$ and $F \leq_B E$, then we call E and F* Borel bireducible *and write $E \sim_B F$.*

If we view Borel equivalence relations as classification problems in the manner described above, then this notion of *Borel reduction* provides a way of comparing their relative "complexities." For instance, if E, F are Borel equivalence relations on X, Y, and if $f : X \to Y$ is a Borel reduction from E to F, then we may think of f as classifying the objects in X up to E by the invariants Y/F, so that in particular the classification problem associated with E is "at most as difficult" as that associated with F in the sense that any set of complete invariants for F works just as well for E, via composition with f. A particular classification problem may be compared with various important benchmarks in the $\leq_B$-partial order so as to determine its relative Borel complexity. For instance, the isomorphism problem for countable graphs mentioned above is, as might be expected, as difficult as it could possibly be in the sense that any equivalence relation of the form $\cong_\sigma$ is Borel reducible to it. (Such a relation is called *Borel complete*). Of course, no one would imagine the the problem of classifying countable graphs up to isomorphism to be easy, but the theory of Borel equivalence relations makes precise just how difficult it is.

Locating particular classification problems within the $\leq_B$-hierarchy is a central problem in Borel equivalence relations, and not surprisingly, there remain interesting examples of classification problems whose Borel complexity has yet to be understood. Recently I have contributed to a joint effort to determine the Borel complexities of various conjugacy problems for the automorphism groups of $\aleph_0$-categorical structures [4]. For example this involves classifying elements of Aut $\mathbb{Q}$, Aut Γ, and $GL(\infty, q)$ up to conjugacy, where $\mathbb{Q}$ is the countable dense linear order, Γ is the random graph, and $V = \mathbb{F}_q^\infty$ is the countably-infinite dimensional vector space over the finite field $\mathbb{F}_q$. In the first two cases the conjugacy problem is Borel complete; however, the second case appears to be related to the important open question of whether isomorphism of countable abelian groups is Borel complete, and thus far we have only succeeded in showing that conjugacy on $GL(V)$ is not Borel. Since a pair of automorphisms U, T of V are conjugate if and only if the corresponding $\mathbb{F}_q[x, x^{-1}]$-modules V_U and V_T are isomorphic as modules, the conjugacy problem quickly leads to questions about the isomorphism problems for modules over a PID.

Research Project 1. *Determine the Borel complexity of conjugacy on* $GL(\infty, q)$*, or at least* $\leq_B$*-compare it with the isomorphism problem for abelian groups. More generally, investigate the extent to which the Borel classification theory of the isomorphism problems for various classes of countable abelian groups generalizes to the corresponding problems for modules over PIDs different from* $\mathbb{Z}$*.*

As another example, it is known that the isomorphism relation on finitely generated groups is extremely complex; however, the complexity of the restriction of this problem to various subclasses of finitely generated groups remains unknown and is of great interest.

Research Project 2. *Determine the Borel complexities of the isomorphism relations on the space of finitely generated amenable groups, and on the space of finitely generated Kazhdan groups.*

In fact, it is still open whether or not these restricted relations admit classification by invariants that are elements in a Polish space.[1] Proving that this is not the case would be a reasonable first step towards understanding their complexities.

Much of my work over the past few years has focused distinguishing relations (under $\leq_B$) inside an important subclass of Borel equivalence relations, namely the *countable* ones, i.e., those with countable equivalence classes. The signature feature of the countable relations is their connection with Borel actions of countable groups. Let us call a standard Borel space X together with a Borel action $G \curvearrowright X$ of a countable group G a *standard Borel* G*-space*, and denote by E_G^X the corresponding G-orbit equivalence relation. Clearly E_G^X is countable Borel, but a remarkable representation theorem of Feldman and Moore [6] asserts the converse: if E is *any* countable Borel equivalence relation on a standard Borel space X, then there is a countable group G and a Borel action $G \curvearrowright X$ such that $E = E_G^X$. Hence distinguishing a pair E, F of countable Borel equivalence relations up to $\leq_B$ essentially amounts to distinguishing the orbit spaces of countable groups.

The fundamental question then becomes: to what extent does E_G^X determine the group G and its action on X? It can be shown that G must act freely and preserve a Borel probability measure μ on X if there is to be any hope of recovering G or its action from E_G^X. Granting this, however,

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[1] Any such equivalence relation is called *smooth*. Specifically, an equivalence relation E on Polish X is *smooth* if there is a Borel reduction from E to $\Delta(Y)$, where $\Delta(Y)$ is the equality relation on some (equivalently any) uncountable standard Borel space Y. By Silver's dichotomy theorem [17], if E is any Borel (in fact coanalytic) equivalence relation with uncountably many equivalence classes, then $\Delta(\mathbb{R}) \leq_B E$. Hence the smooth relations are $\leq_B$-least Borel equivalence relations.

and under certain (rather strong) hypotheses on G and its action, it turns out that from the Borel complexity of E_G^X alone one may indeed recover a significant amount of information about G and the action $G \curvearrowright X$. This phenomenon is referred to as *Borel superrigidity.*

In [19] Thomas proved a Borel superrigidity theorem for certain actions of lattices in higher-rank simple Lie groups, and used it to prove that for $n \geq 3$, the orbit equivalence relations arising from the natural actions of $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$, and of $SL_n(\mathbb{Z})$ on n-dimensional p-adic projective space, are pairwise $\leq_B$-incomparable as p varies. The Borel superrigidity theorem of [19] is based on Furman's orbit equivalence rigidity theorem [9], which unfortunately fails for the low-rank Lie group $SL_2(\mathbb{R})$, and hence analogues of the theorems in [19] remain open for $n = 2$. In [20], Thomas extended the results of [19] in the direction of $n = 2$ by showing how to apply Zimmer superrigidity theory [22] to actions (on $SL_2(\mathbb{Z}_p)$ and on p-adic projective lines) of groups of the form $SL_2(\mathbb{Z}[\frac{1}{q}])$. Subsequently I did the same in [15] for groups of the form $SL_2(\mathbb{Z}[\sqrt{q}\,])$. However, it remains unclear how best to approach the corresponding problem for $SL_2(\mathbb{Z})$. Specifically, we would like to show the following:

Conjecture 2. *Let E_p be the orbit equivalence relation arising from the natural action of $SL_2(\mathbb{Z})$ on $SL_2(\mathbb{Z}_p)$. Then E_p and E_q are $\leq_B$-incomparable whenever $p \neq q$ are distinct primes.*

Conjecture 3. *Let F_p be the orbit equivalence relation arising from the natural action of $SL_2(\mathbb{Z})$ on the p-adic projective line, $PG(1, \mathbb{Q}_p)$. Then F_p and F_q are $\leq_B$-incomparable whenever $p \neq q$ are distinct primes.*

Since $SL_2(\mathbb{Z})$ contains a finite index nonabelian free subgroup, $E_{SL_2(\mathbb{Z})}^X$ is *treeable* whenever X is a free standard Borel $SL_2(\mathbb{Z})$-space.[2] Therefore establishing Conjectures 2 and 3 would provide for the first time concrete examples of infinite, pairwise $\leq_B$-incomparable families of treeable countable Borel equivalence relations.

$SL_2(\mathbb{Z})$ also lies beyond the grasp of a powerful recent superrigidity result of Popa's concerning Bernoulli actions. Roughly speaking, Popa's theorem says that for any Bernoulli action $\Gamma \curvearrowright X^\Gamma$ of a Kazhdan group Γ, and for *any* countable discrete group Λ, every cocycle $\alpha : \Gamma \times X \rightarrow \Lambda$ is equivalent to a homomorphism $\Gamma \rightarrow \Lambda$. The strength of Popa's superrigidity theorem is unprecedented in that there is no assumption on the discrete target group Λ, no assumption on the cocycle α, and relatively mild assumptions on the source group Γ.[3] In [21], Thomas used Popa superrigidity to answer a number of important open questions, showing, for instance, that the isomorphism relation on finite-rank torsion-free abelian groups is not countable universal, and that not every countable

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[2] A countable Borel equivalence relation E on a standard Borel space X is said to be *treeable* if there is a Borel acyclic graph on X whose connected components are precisely the E-classes.

[3] In fact the most general version of Popa's theorem requires only that Γ is nonamenable with infinite center.

Borel equivalence relation is Borel bireducible with the orbit equivalence relation of a *free* Borel group action.[4] Thomas's results, however, use only a fraction of the full strength of Popa's theorem, and it seems quite plausible that further applications of the theorem await discovery.

Research Project 3. *Find further applications of Popa's superrigidity results to countable Borel equivalence relations.*

Unfortunately, as alluded to above, even the most general form of Popa's theorem (see [14]) cannot be applied to subgroups of $SL_2(\mathbb{C})$, illustrating yet again the theme that as yet we have no methods for dealing with $SL_2(\mathbb{Z})$-actions. Using the techniques of [20] and [15], I was able to prove that Bernoulli actions of $PSL_2(\mathbb{Z}[\frac{1}{p}])$, and also of $PSL_2(\mathbb{Z}[\sqrt{p}\,])$, are pairwise $\leq_B$-incomparable as p varies. However, these methods still depend on Zimmer superrigidity, which cannot be applied to actions of the low-rank lattice $SL_2(\mathbb{Z})$.

Research Project 4. *Continue to investigate methods for analyzing $SL_2(\mathbb{Z})$-actions, and in particular work towards proving Conjectures 2 and 3. More generally, work towards developing methods for distinguishing treeable countable Borel equivalence relations up to $\leq_B$.*

Since this project is in all likelihood too difficult to view as anything more than a distant goal, let me indicate some more modest goals that might help get the project under way. For instance, it seems reasonable to analyze the continuous functions from $SL_2(\mathbb{Z}_p)$ to $SL_2(\mathbb{Z}_q)$, $p \neq q$, with the goal of proving the weaker result that there is no *continuous* reduction from E_p to E_q. In fact, it would be worthwhile to investigate the existence of continuous reductions more generally.

Research Project 5. *Show that there is no continuous reduction from E_p to E_q. Or, find naturally occurring pairs of countable Borel equivalence relations $E \leq_B F$ that admit no continuous reduction from E to F.*

Yet another line of approach towards understanding E_p might be to examine its subrelations, especially those generated by the actions of the principal congruence subgroups of $SL_2(\mathbb{Z})$. Thomas has shown that for $n \geq 3$, if $X = SL_n(\mathbb{Z}_p)$ and $\Lambda = \ker\,\psi \leq SL_n(\mathbb{Z})$, where

$$\psi : SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/p\mathbb{Z})$$

is the canonical surjection, then $E^X_{SL_n(\mathbb{Z})} <_B E^X_\Lambda$. Unfortunately this result makes essential use of the fact that for $n \geq 3$, $SL_n(\mathbb{Z})$ is Kazhdan; however, it suggests that one should investigate the analogous situation for $n = 2$. A starting point here might be work of Feldman, Sutherland, and

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[4] Here a countable Borel equivalence relation E is said to be *universal* if $F \leq_B E$ for every countable Borel equivalence relation F. It can easily be shown that the orbit equivalence relation E_∞ arising from the shift action of the two-generator free group $\mathbb{F}_2$ on its power set is universal, as is any other countable Borel equivalence relation that is Borel bireducible with it.

Zimmer [7] characterizing the normal ergodic subrelations of, e.g., $E_{SL_n(\mathbb{Z})}^{SL_n(\mathbb{Z}_p)}$ for $n \geq 3$. Of course, the action of a principal congruence subgroup of $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$ will not be ergodic, so one should investigate [7, Theorem 4.1] in the absence of ergodicity.

Research Project 6. *Using* [7]*, attempt to classify the normal subrelations of* $E_{SL_3(\mathbb{Z})}^{SL_3(\mathbb{Z}_p)}$ *up to orbit equivalence or Borel reducibility.*

To mention one last project concerning linear actions on projective spaces, consider the natural action of $GL_2(\mathbb{Q})$ on the real projective line $PG(1,\mathbb{R})$. It is well known that the orbit equivalence relation arising from the restricted action of $SL_2(\mathbb{Z})$ on $PG(1,\mathbb{R})$ is hyperfinite, but it would be interesting to know whether this holds as well for the coarser relation.[5]

Conjecture 4. *The orbit equivalence relation arising from* $GL_2(\mathbb{Q}) \curvearrowright PG(1,\mathbb{R})$ *is hyperfinite.*

Recently I have also been involved in developing a theory of "Borel invariant properties" of countable Borel equivalence relations, the motivating example of which is *Borel boundedness* [2]. The countable Borel equivalence relation E on X is said to be *Borel bounded* if for every Borel function $\phi : X \to \omega^\omega$, there exists a Borel homormophism $\psi : X \to \omega^\omega$ from E to $=^*$ such that $\psi(x) \leq^* \phi(x)$ for every $x \in X$, where '$*$' means "up to finite error." In [2], Boykin and Jackson showed that Borel boundedness is closely related to the "unions problem" by proving that a countable increasing union of hyperfinite Borel equivalence relations is hyperfinite if and only if it is Borel bounded.

The definition of Borel boundedness is based on the fact that the bounding number, $\mathfrak{b}$, is uncountable; ie, no countable family $\mathcal{F} \subseteq \omega^\omega$ can be bounded simultaneously by a single function $\alpha \in \omega^\omega$. Building upon this example, Coskey and I introduce in [3] new "Borel invariant" properties of countable Borel equivalence relations that correspond to various other cardinal invariants of the continuum in the same way that Borel boundedness corresponds to $\mathfrak{b}$. We develop the basic facts about these properties and establish some relationships between them, but it is clear that further work can be done.

Research Project 7. *Continue to investigate the Borel cardinal invariant properties introduced in* [3]*, and in particular determine whether any of these properties provides additional insight into the unions problem or the related problem concerning tail equivalence relations (see* [12, 6.1E]*).*

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[5] A countable Borel equivalence relation E on a standard Borel space X is said to be *hyperfinite* if it can be written as an increasing union of *finite* equivalence relations, i.e., those with finitely many classes. Equivalently, E is hyperfinite *iff* $E = E_{\mathbb{Z}}^X$ for some standard Borel $\mathbb{Z}$-space X *iff* $E \leq_B E_0$. By a dichotomy theorem of Harrington-Kechris-Louveau [10], $E_0 \leq_B E$ for any nonsmooth countable Borel equivalence relation E. In fact, E_0 is the only nonsmooth hyperfinite countable Borel equivalence relation up to $\sim_B$ [5].

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