RESEARCH PROPOSAL: BOREL EQUIVALENCE RELATIONS

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The last twenty years have seen the emergence of a new direction of research in descriptive set theory, namely the study of definable equivalence relations on complete separable metric spaces (i.e., *Polish spaces*) and their resulting quotients. Such quotients include the orbit space of an ergodic group action, the unitary dual of a Polish group, the space of measure classes of Borel probability measures on a Polish space, the set of Turing degrees of subsets of natural numbers, and the space of isomorphism classes of countable models of some $\mathcal{L}_{\omega_1,\omega}$-sentence, to name a few. Of course, for all but the simplest equivalence relations E on a Polish space X, the collection X/E of E-classes cannot be viewed in any reasonable way as a definable set inside a Polish space, and so the usual descriptive techniques of topology, category, measure theory, etc., do not apply in the usual way to their study. The emerging field of Borel equivalence relations seeks to understand the structure of these "singular" spaces X/E. Its methods, as they have been developed over the past decade, draw upon diverse areas of mathematics such as model theory, topology, ergodic theory, measurable dynamics and orbit equivalence theory, as well as the theory of various classes of groups and their definable actions, including especially free groups, amenable groups, Kazhdan groups, Polish groups, and Lie groups together with their lattice subgroups.

One reason for studying the moduli spaces X/E is that they appear as sets of invariants for classification problems arising throughout mathematics. A *standard Borel space* is a Polish space equipped only with its σ-algebra of Borel sets. (By a classical result of Kuratowski, there is only one uncountable such space up to isomorphism). A wide range of naturally occurring classes of mathematical objects may be given the structure of a standard Borel space, and in many of these cases the natural notions of classification turn out to be definable equivalence relations on that space. Consider, for instance, the problem of classifying countable graphs up to isomorphism. Letting $\mathcal{C}$ be the set of graphs of the form $\Gamma = \langle \mathbb{N}, E\rangle$ and identifying each graph $\Gamma \in \mathcal{C}$ with its edge relation $E \in 2^{\mathbb{N}^2}$, one may check that $\mathcal{C}$ is a Borel subset of the Polish space $2^{\mathbb{N}^2}$, and hence is itself a standard Borel space. Moreover, the isomorphism relation on $\mathcal{C}$ is simply the orbit equivalence relation arising from the natural action of the infinite symmetric group $\mathrm{Sym}(\mathbb{N})$ on $\mathcal{C}$. More generally, if σ is any $\mathcal{L}_{\omega_1,\omega}$ sentence of a countable language $\mathcal{L}$, then

$$\mathrm{Mod}(\sigma) = \{\mathcal{M} = \langle \mathbb{N}, \cdots \rangle \mid \mathcal{M} \models \sigma\}$$

is a standard Borel space, and the isomorphism relation on $\mathrm{Mod}(\sigma)$ is the orbit equivalence relation generated by the $\mathrm{Sym}(\mathbb{N})$-action. It can be shown that this orbit equivalence relation is always analytic as a subset of $\mathrm{Mod}(\sigma)\times\mathrm{Mod}(\sigma)$, and that if the collection of objects defined by σ is "finitely generated" in some broad sense then it will in fact be Borel. (For instance, the graph isomorphism relation restricted to countable, connected, *locally finite* graphs is Borel). With these examples in mind we make the following definitions.

Definition 1. *An equivalence relation $E \subseteq X \times X$ on a standard Borel space X is called* Borel *(*analytic*, etc.) iff it is Borel as a subset of $X \times X$. If E and F are Borel equivalence relations on the standard Borel spaces X, Y respectively, then we call a Borel function $f : X \to Y$ a* Borel reduction *from E to F if for all $x, y \in X$,*

$$x \, E \, y \Longleftrightarrow f(x) \, F \, f(y).$$

If such a function exists then we write

$$E \leq_B F$$

and say that E is Borel reducible *to F. We write*

$$\begin{aligned} E <_B F &\iff E \leq_B F \text{ and } F \not\leq_B E, \\ E \sim_B F &\iff E \leq_B F \text{ and } F \leq_B E. \end{aligned}$$

If $E \sim_B F$, then we call E and F Borel bireducible.

If we view Borel equivalence relations as classification problems in the manner described above, then this notion of *Borel reduction* provides a way of comparing their relative "complexities." For instance, if E and F are Borel equivalence relations on the standard Borel spaces X and Y, respectively, and if $f : X \to Y$ is a Borel reduction from E to F, then we may think of f as classifying the objects in X up to E by the invariants Y/F, so that in particular the classification problem associated with E is "at most as difficult" as that associated with F in the sense that any set of complete invariants for F works just as well for E, via composition with f. A particular classification problem may be compared with various important benchmarks in the $\leq_B$-partial order so as to determine its relative Borel complexity. For instance, the isomorphism problem for countable graphs mentioned above is, as might be expected, as difficult as it could possibly be in the sense that any other $\mathrm{Sym}(\mathbb{N})$-orbit equivalence relation on some standard Borel space $\mathrm{Mod}(\sigma)$ Borel reduces to it. Of course, no one would imagine the problem of classifying countable graphs up to isomorphism to be easy; but the theory of Borel equivalence relations makes precise just how difficult it is.

Locating particular classification problems within the $\leq_B$-hierarchy is a central problem in Borel equivalence relations; and not surprisingly, there remain interesting examples of classification problems whose Borel complexity has yet to be understood. For instance, it is known that the isomorphism relation on finitely generated groups is extremely complex (in fact, bireducible with isomorphism of connected locally finite graphs). However, the complexity of the restriction of this problem to various subclasses of finitely generated groups remains unknown and is of great interest.

Research Project 1. *Determine the Borel complexity of the isomorphism relation on the space of finitely generated amenable groups, and on the space of finitely generated Kazhdan groups.*

In fact, it is still open whether or not these restricted relations admit classification by invariants that are elements in a Polish space.[1] Proving this is not the case would be a reasonable first step towards understanding their complexities.

My own work over the past two years has focused on employing existing techniques and developing new ones to distinguish relations (with respect to $\leq_B$) inside an important subclass of Borel equivalence relations, namely the *countable* ones, i.e., those with countable equivalence classes. The signature feature of the countable relations is their connection with Borel actions of countable groups. Let us call a standard Borel space X together with a Borel action $G \curvearrowright X$ of a countable group G a *standard Borel G-space*, and denote by E_G^X the corresponding G-orbit equivalence relation. It is clear that E_G^X is countable Borel, but a remarkable theorem of Feldman and Moore [6] asserts the converse: if E is *any* countable Borel equivalence relation on a standard Bore space X, then there is a countable group G and a Borel action $G \curvearrowright X$ such that $E = E_G^X$. Hence distinguishing a pair E, F of countable Borel equivalence relations up to $\leq_B$ essentially amounts to distinguishing the orbit spaces of countable groups.

The fundamental question then becomes: to what extent does E_G^X determine the group G and its action on X? It can be shown that G must act freely and preserve a Borel probability measure μ on X if there is to be any hope of recovering G or its action from E_G^X. Granting this, however, and under certain (rather strong) hypotheses on G and its action, it turns out that from the Borel complexity of E_G^X alone one may indeed recover a significant amount of information about G and the action $G \curvearrowright X$. This phenomenon is referred to as *Borel superrigidity.* As an example we cite the Borel superrigidity theorem that appears in Thomas [26]. Suppose Γ_1 and Γ_2 are lattices in a connected, centerless, simple Lie group G of rank at least 2, and for each each $i = 1, 2$, suppose X_i is a free standard Borel Γ_i-space with invariant, ergodic, nonatomic probability measure μ_i. Under

[1] Any such equivalence relation is called *smooth.* Specifically, an equivalence relation E on Polish X is *smooth* iff there is a Borel reduction from E to $\Delta(Y)$, where $\Delta(Y)$ is the equality relation on some (equivalently any) uncountable standard Borel space Y. By Silver's dichotomy theorem [24], if E is any Borel (in fact coanalytic) equivalence relation with uncountably many equivalence classes, then $\Delta(\mathbb{R}) \leq_B E$. Hence the smooth relations are $\leq_B$-least Borel equivalence relations.

some additional (and reasonably mild) technical hypotheses on the actions, we have the following: if $E_{\Gamma_1}^{X_1} \leq_B E_{\Gamma_2}^{X_2}$, then (X_1, Γ_1, μ_1) and (X_2, Γ_2, μ_2) are virtually isomorphic; in particular Γ_1 and Γ_2 have isomorphic finite index subgroups.

In [26] Thomas used this to prove that for $n \geq 3$, the orbit equivalence relations arising from the natural actions of $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$, and of $SL_n(\mathbb{Z})$ on n-dimensional p-adic projective space, are pairwise $\leq_B$-incomparable as p varies. The Borel superrigidity theorem of [26] is based on Furman's orbit equivalence rigidity theorem [9], which unfortunately fails for the low-rank Lie group $SL_2(\mathbb{R})$, and hence the theorems of [26] remain open for $n = 2$. In [27], Thomas extended the results of [26] in the direction of $n = 2$ by showing how to apply Zimmer superrigidity theory [29] to actions (on $SL_2(\mathbb{Z}_p)$ and on p-adic projective lines) of groups of the form $SL_2(\mathbb{Z}[\frac{1}{q}])$. Subsequently I did the same in [21] for groups of the form $SL_2(\mathbb{Z}[\sqrt{q}])$. However, it remains unclear how best to approach the corresponding problem for $SL_2(\mathbb{Z})$. Specifically, we would like to show the following:

Conjecture 2. *Let E_p be the orbit equivalence relation arising from the natural action of $SL_2(\mathbb{Z})$ on $SL_2(\mathbb{Z}_p)$. Then E_p and E_q are $\leq_B$-incomparable whenever $p \neq q$ are distinct primes.*

Conjecture 3. *Let F_p be the orbit equivalence relation arising from the natural action of $SL_2(\mathbb{Z})$ on the p-adic projective line, $PG(1, \mathbb{Q}_p)$. Then F_p and F_q are $\leq_B$-incomparable whenever $p \neq q$ are distinct primes.*

Establishing the above conjectures would be highly desirable for the following reason. Since $SL_2(\mathbb{Z})$ contains a finite index nonabelian free subgroup, it follows that $E_{SL_2(\mathbb{Z})}^{X}$ is *treeable* whenever X is a free standard Borel $SL_2(\mathbb{Z})$-space.[2] But at present only three $\sim_B$-distinct nonsmooth treeable relations are known, and it is an important open question whether there exist infinitely many $\leq_B$-incomparable, or even $\sim_B$-distinct, treeable countable Borel equivalence relations. [Addendum: Greg Hjorth has just shown that there exist infinitely many pairwise $\sim_B$-distinct treeable relations; however, no concrete examples are yet known. In particular, the above conjectures remain open.]

$SL_2(\mathbb{Z})$ also lies beyond the grasp of a powerful recent superrigidity result of Popa's concerning Bernoulli actions. Roughly speaking, Popa's theorem says that for any Bernoulli action $\Gamma \curvearrowright X^{\Gamma}$ of a Kazhdan group Γ, and for *any* countable discrete group Λ, every cocycle $\alpha : \Gamma \times X \to \Lambda$ is equivalent to a homomorphism $\Gamma \to \Lambda$. The strength of Popa's superrigidity theorem is unprecedented in that there is *no* assumption on the discrete target group Λ, no assumption on the cocycle α, and relatively mild assumptions on the source group Γ.[3] In [28], Thomas used Popa superrigidity to answer a number of important open questions, showing, for instance, that the isomorphism relation

[2] A countable Borel equivalence relation E on a standard Borel space X is said to be *treeable* iff there is a Borel acyclic graph on X whose connected components are precisely the E-classes.

[3] In fact the most general version of Popa's theorem requires only that Γ is nonamenable with infinite center.

on finite-rank torsion-free abelian groups is not countable universal, and that not every countable Borel equivalence relation is Borel bireducible with the orbit equivalence relation of a *free* Borel group action.[4] Thomas's results, however, use only a fraction of the full strength of Popa's theorem, and it seems quite plausible that further applications of the theorem await discovery.

Research Project 2. *Find further applications of Popa's superrigidity results to countable Borel equivalence relations.*

Unfortunately, as alluded to above, even the most general form of Popa's theorem (see [20]) cannot be applied to subgroups of $SL_2(\mathbb{C})$, illustrating yet again the theme that as yet we have no methods for dealing with $SL_2(\mathbb{Z})$-actions. Using the techniques of [27] and [21], I was able to prove that Bernoulli actions of $PSL_2(\mathbb{Z}[\frac{1}{p}])$, and also of $PSL_2(\mathbb{Z}[\sqrt{p}\,])$, are pairwise $\leq_B$-incomparable as p varies. However, these methods still depend on Zimmer superrigidity, which cannot be applied to actions of the low-rank lattice $SL_2(\mathbb{Z})$.

Research Project 3. *Continue to investigate methods for analyzing $SL_2(\mathbb{Z})$-actions, and in particular work towards proving Conjectures 2 and 3. More generally, work towards developing methods for distinguishing treeable countable Borel equivalence relations up to $\leq_B$.*

Since this project is in all likelihood too difficult to view as anything more than a distant goal, let me indicate some more modest goals that might help get the project under way. For instance, it seems reasonable to analyze the continuous functions from $SL_2(\mathbb{Z}_p)$ to $SL_2(\mathbb{Z}_q)$, $p \neq q$, with the goal of proving the weaker result that there is no *continuous* reduction from E_p to E_q. In fact, it should be worthwhile to investigate the existence of continuous reductions more generally; for instance, while it is not too difficult to construct *ad hoc* examples, I am not aware of a naturally occurring pair $E \leq_B F$ of countable Borel equivalence relations such that there is no *continuous* reduction from E to F. Thomas has suggested that one such pair might consist of the commensurability and isomorphism relations on the space of finitely generated groups.

Research Project 4. *Show that there is no continuous reduction from E_p to E_q. Or, find any naturally occurring pair of countable Borel equivalence relations $E \leq_B F$ such that there is no continuous reduction from E to F.*

Yet another line of approach towards understanding E_p might be to examine its subrelations, especially those generated by the actions of the principal congruence subgroups of $SL_2(\mathbb{Z})$. Thomas

[4] Here a countable Borel equivalence relation E is said to be *universal* iff $F \leq_B E$ for every countable Borel equivalence relation F. It can easily be shown that the orbit equivalence relation E_∞ arising from the shift action of the two-generator free group $\mathbb{F}_2$ on its power set is universal, as is any other countable Borel equivalence relation that is Borel bireducible with it.

has shown that for $n \geq 3$, if $X = SL_n(\mathbb{Z}_p)$ and $\Lambda = \ker \psi \leq SL_n(\mathbb{Z})$, where

$$\psi : SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/p\mathbb{Z})$$

is the canonical surjection, then $E^X_{SL_n(\mathbb{Z})} <_B E^X_\Lambda$. Unfortunately this result makes essential use of the fact that for $n \geq 3$, $SL_n(\mathbb{Z})$ is Kazhdan; however, it suggests that one should investigate the analogous situation for $n = 2$. A starting point here might be work of Feldman, Sutherland, and Zimmer [7] characterizing the normal ergodic subrelations of, e.g., $E^{SL_n(\mathbb{Z}_p)}_{SL_n(\mathbb{Z})}$ for $n \geq 3$. Of course, the action of a principal congruence subgroup of $SL_n(\mathbb{Z})$ on $SL_n(\mathbb{Z}_p)$ will not be ergodic, so one should investigate [7, Theorem 4.1] in the absence of ergodicity.

Research Project 5. *Using* [7]*, attempt to classify the normal subrelations of* $E^{SL_3(\mathbb{Z}_p)}_{SL_3(\mathbb{Z})}$ *up to orbit equivalence or Borel reducibility.*

To mention one last project concerning linear actions on projective spaces, consider the natural action of $GL_2(\mathbb{Q})$ on the real projective line $PG(1, \mathbb{R})$. It is well known that the orbit equivalence relation arising from the restricted action of $SL_2(\mathbb{Z})$ on $PG(1, \mathbb{R})$ is hyperfinite, but it would be interesting to know whether this holds as well for the coarser relation.[5]

Conjecture 4. *The orbit equivalence relation arising from* $GL_2(\mathbb{Q}) \curvearrowright PG(1, \mathbb{R})$ *is hyperfinite.*

Let me return now to the main technique available for distinguishing countable Borel equivalence relations up to Borel reducibility: namely, superrigidity. Many of the results mentioned thus far depend upon Zimmer's cocycle superrigidity theorems [29, 5.2.5 and 10.6.1], the related strong orbit rigidity theorem of Furman [9], or Popa's recent superrigidity theorem concerning Bernoulli actions [19]. But in fact additional examples of superrigidity are available, some quite recent: for example, Monod and Shalom have proven orbit equivalence superrigidity results for actions of products of word-hyperbolic groups [16], [17]; Kida has shown orbit equivalence superrigidity for ergodic actions of the mapping class groups [14]; and Ioana has proven a cocycle superrigidity theorem for profinite actions of Kazhdan groups [12]. It seems likely that these theorems will yield further applications to the theory of Borel equivalence relations. We have already mentioned that Popa's deep theorems have not yet been exploited in their full generality; and similarly, some initial applications of Ioana superrigidity ([4], [22, 4.3.1]) suggest that further study might pay dividends. To cite a specific (and presumably quite tractable) example, Shalom has recently shown that $SL_5(\mathbb{Z}[x])$ is Kazhdan

[5]A countable Borel equivalence relation E on a standard Borel space X is said to be *hyperfinite* iff it can be written as an increasing union of *finite* equivalence relations, i.e., those with finitely many classes. Equivalently, E is hyperfinite *iff* $E = E^X_{\mathbb{Z}}$ for some standard Borel $\mathbb{Z}$-space X *iff* $E \leq_B E_0$. By a dichotomy theorem of Harrington-Kechris-Louveau [10], $E_0 \leq_B E$ for any nonsmooth countable Borel equivalence relation E. In fact, E_0 is the only nonsmooth hyperfinite countable Borel equivalence relation up to $\sim_B$ [5].

[23], and hence for J an infinite set of primes, the profinite actions

$$SL_5(\mathbb{Z}[x]) \curvearrowright \prod_{p \in J} SL_5(\mathbb{F}_{p^{r(p)}})$$

would appear to fall within the context of Ioana's theorem.

Furthermore, thus far there have been no applications of Monod-Shalom or Kida superrigidity to Borel equivalence relations. By Kida [14, 1.1], any essentially free, probability measure-preserving ergodic action of a higher-complexity mapping class group is orbit equivalence superrigid; and presumably this should cover, for instance, the essentially free ergodic actions of mapping class groups on spaces of measured foliations described in [15].

Research Project 6. *Look for new applications of the various existing superrigidity results mentioned above to countable Borel equivalence relations.*

To close this proposal, I would like to mention several basic (but possibly quite difficult) open problems that any research programme in this field should keep in mind at all times.

One collection of open questions concerns Turing equivalence, Martin's Conjecture, and the following conjecture of Hjorth's: if $E \subseteq F$ are countable Borel equivalence relations and E is universal, then F is also universal. Adams [1] has constructed countable Borel equivalence relations $E \subseteq F$ that are $\leq_B$-incomparable, and Thomas [26] has constructed a pair $E \subseteq F$ such that $F <_B E$, but the question of whether a relation containing a universal subrelation is necessarily universal remains open. As noted in [3], if this were true then it would imply the universality of $\equiv_T$ and refute the well known Martin Conjecture on degree invariant Borel maps.[6]

Another basic problem concerns amenability and hyperfinite relations. It is an important open question to determine those groups G for which E_X^G is always hyperfinite. It is known that for every nonamenable countable group G there is a Borel action $G \curvearrowright X$ such that E_G^X is *not* hyperfinite, and it is known that if G is countable abelian, then E_G^X must be hyperfinite [13]. Conjecturally, E_G^X is hyperfinite whenever G is a countable amenable group, but at present the best known result is that if G is countable amenable and X is a standard Borel G-space, then for every Borel probability measure μ on X, there is μ-conull $Y \subseteq X$ such that $E_G^X \restriction Y$ is hyperfinite [18].

Finally, and somewhat incredibly, no example is known of a non-smooth countable Borel equivalence relation that has an immediate $\leq_B$-successor; and no example is known of a non-universal countable Borel equivalence relation that does *not* have an immediate $\leq_B$-successor.

[6] Here $\equiv_T$ is the relation of Turing equivalence of subsets of natural numbers. It is an important open question whether or not $\equiv_T$ is universal.

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