

WORKSHOP

Work through the following problems together with the other members of your group, in the class time provided for you. Use books, notes, or any resource you can think of. Then write up formal solutions to the problems and turn them in next Monday, December 7th. The Workshop will count towards your quiz grade, and will count as two quizzes (for a total of 20 points). Each problem, however, is worth 4 points, giving you an opportunity to earn a total of 28 points out of 20. Any points beyond 20 will be tacked on to your quiz grade as extra credit points. Remember to include brief explanations of your reasoning along with your answers!

- (1) Recall that if X and Y are sets, then $X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$ is the intersection of X and Y . It can easily be shown that if U and V are subspaces of \mathbb{R}^n , then the intersection $U \cap V$ is also a subspace of \mathbb{R}^n .

$$\text{Let } u_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, v_1 = \begin{bmatrix} 3 \\ -2 \\ 3 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}.$$

Let $U = \text{Span}\{u_1, u_2, u_3\}$, and $V = \text{Span}\{v_1, v_2, v_3\}$, so that U and V are 3-dimensional subspaces of \mathbb{R}^4 . Find a basis for the subspace $U \cap V$.

- (2) The three points

$$P(-\sqrt{18}, \sqrt{12}, \sqrt{72}, 6, \sqrt{54}), \quad Q(-\sqrt{8}, 0, \sqrt{32}, 0, \sqrt{24}), \quad \text{and} \quad R(-\sqrt{8}, \sqrt{48}, \sqrt{32}, 12, \sqrt{24})$$

determine a plane (ie, a 2-dimensional subspace) in \mathbb{R}^5 , and inside this plane the points are successive vertices of a regular polygon. How many sides does this polygon have?

- (3) Let $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \right\}$, so that \mathcal{B} and \mathcal{C} are bases for \mathbb{R}^3 . Find a 3×3 matrix A such that $A(v)_{\mathcal{B}} = (v)_{\mathcal{C}}$ for all $v \in \mathbb{R}^3$. (Recall that for $v \in \mathbb{R}^3$, $(v)_{\mathcal{B}}$ is the coordinate vector of v relative to the basis \mathcal{B}).

- (4) Let W be the subspace of \mathbb{R}^5 with basis $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 3 \end{bmatrix} \right\}$. Find a square matrix A whose null space is W .

- (5) An *orthogonal basis* for a subspace V of \mathbb{R}^n is a basis \mathcal{B} for V such that any pair of vectors $u, v \in \mathcal{B}$ are orthogonal. Find an orthogonal basis for \mathbb{R}^4 that contains as subsets orthogonal bases for $\text{Col } A$ and for $\text{Null } A^T$,

$$\text{where } A = \begin{bmatrix} 1 & 2 & -1 & -1 \\ 1 & -1 & 5 & 2 \\ 1 & -1 & 5 & 2 \\ -1 & 1 & -5 & -2 \end{bmatrix}.$$

- (6) We know that the RREF of a matrix is unique; given any matrix A , denote the RREF of A by R_A . Then, given R_A , we may further “reduce” A by finding the RREF of the *transpose* of R_A , which, of course, will also be unique. Thus for any matrix A , let S_A denote the RREF of R_A^T . So to any $m \times n$ matrix A we can associate the $n \times m$ matrix S_A . Now, as A ranges over the set of all $m \times n$ matrices, how many distinct $n \times m$ matrices S_A are there?
- (7) Suppose that U , V , and W are subspaces of \mathbb{R}^{40} of dimensions 32, 37, and 30 respectively. How many different values for $\dim U \cap V \cap W$ are consistent with this information?