FINAL EXAM PRACTICE PROBLEMS

(1) Let $R = \begin{bmatrix} 1 & 0 & -1 & 3 & 9 \\ 0 & 1 & 2 & -5 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Suppose first that R is the RREF of the augmented matrix of a linear system, and express the solution to the system parametrically. Then find bases for Col R, Row R, and Null R.

(2) Determine if the linear system

$$\begin{cases} 2x_1 - 2x_2 + 4x_3 = 1 \\ -4x_1 + 4x_2 - 8x_3 = -3 \end{cases}$$

is consistent, and if so find its general solution.

(3) Is the vector $\vec{v} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$ in the span of the set $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$? If so, express $\vec{v}$ as a linear combination of the vectors in S.

(4) Is the set $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$ linearly independent?

(5) Determine if the matrix $\begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ is invertible, and, if so, find its inverse.

(6) Find the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, and use it to solve the system $\begin{cases} x_1 + 2x_2 = 9 \\ 2x_1 + 3x_2 = 3 \end{cases}$

(7) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \end{bmatrix}$. Find the reduced row echelon form R of A and an invertible matrix P such that $PA = R$.

(8) Find a spanning set for the subspace $\left\{ \begin{bmatrix} 2s - 5t \\ 3r + s - 2t \\ r - 4s + 3t \\ -r + 2s \end{bmatrix} \in \mathbb{R}^4 : r, s, t \in \mathbb{R} \right\}$.

(9) Find a spanning set for the null space of the matrix $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$.

(10) Find a spanning set for the range of the linear transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 \\ x_2 \end{bmatrix}$.

(11) Find the determinants of $A = \begin{bmatrix} 8 & 0 & 0 \\ -1 & -2 & 0 \\ 4 & 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 2 & 0 \\ 6 & -4 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} -2 & 1 & -2 \\ 4 & -2 & -1 \\ 0 & 3 & 6 \end{bmatrix}$.

(12) Find the unit vector in the direction opposite to $\vec{v} = \langle -4, 4, 2 \rangle$.

(13) Find parametric equations of the line perpendicular to the xy-plane passing through the origin.

(14) Determine if the lines $\vec{r}_1(t) = \langle 0, 1, 1 \rangle + t\langle 1, 1, 2 \rangle$ and $\vec{r}_2(s) = \langle 2, 0, 2 \rangle + s\langle 1, 4, 4 \rangle$ intersect, and if so, find the point of intersection.

(15) Determine whether the two vectors $\langle 1, 1, 1 \rangle$ and $\langle 1, -2, -2 \rangle$ are orthogonal, and if not, whether the angle between them is acute or obtuse.

(16) Find an equation of the plane passing through the three points $P = (2, -1, 4)$, $Q = (1, 1, 1)$, and $R = (3, 1, -2)$.

(17) Find two unit vectors orthogonal to both $\vec{a} = \langle 3, 1, 1 \rangle$ and $\vec{b} = \langle -1, 2, 1 \rangle$.

(18) Give an example of a matrix A whose column space and null space are equal.

(19) Let A be an invertible $n \times n$ matrix. Find a necessary and sufficient condition for $\det A$ to equal $\det(-A)$.

(20) Suppose A is an $n \times n$ matrix such that A^4 is the $n \times n$ zero matrix. Is this enough information to determine whether or not A is invertible? If so, is A invertible?

(21) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^2$, with $\text{Null } T = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_1 = x_2 \right\}$. Is this enough information to determine whether or not T is onto? If so, is T onto?

(22) Give an example of a subset of $\mathbb{R}^2$ that is closed under vector addition but is *not* closed under scalar multiplication.

(23) Give an example of a subset of $\mathbb{R}^2$ that is closed under scalar multiplication but is *not* closed under vector addition.

(24) Does there exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}?$$

(25) Let V be the subspace of $\mathbb{R}^5$ defined by

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbb{R}^5 : x_3 = 5x_4 \right\}.$$

Find a basis for V.

(26) Let U and V be subspaces of $\mathbb{R}^n$, and let $W = U \cup V$ be the union of U and V, ie, W is the set of all vectors in $\mathbb{R}^n$ that belong to either U or V. In symbols,

$$W = \{x \in \mathbb{R}^n : x \in U \text{ or } x \in V\}.$$

Is it *possible* that W is a subspace of $\mathbb{R}^n$? *Must* it be true that W is a subspace of $\mathbb{R}^n$?

(27) Express the invertible matrix $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ as a product of elementary matrices.

(28) Give an example of an $n \times n$ matrix A with linearly independent columns and linearly dependent rows, or explain why this is impossible.

(29) Writing $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in each case below, determine which of the following are subspaces of $\mathbb{R}^3$:

(a) $W_1 = \{x \in \mathbb{R}^3 : x_1 = 3x_2 \text{ and } x_3 = -x_2\}$
(b) $W_2 = \{x \in \mathbb{R}^3 : x_1 = x_3 + 2\}$
(c) $W_3 = \{x \in \mathbb{R}^3 : 2x_1 - 7x_2 + x_3 = 0\}$
(d) $W_4 = \{x \in \mathbb{R}^3 : x_1 + 2x_2 - 3x_3 = 1\}$
(e) $W_5 = \{x \in \mathbb{R}^3 : 5x_1^2 - 3x_2^2 + 6x_3^2 = 0\}$

(30) Suppose $T : \mathbb{R}^8 \to \mathbb{R}^5$ is a linear transformation, and suppose that $\mathcal{B}$ is a basis for $\mathbb{R}^5$. Suppose further that every vector in $\mathcal{B}$ is in the range of T. Is this enough information to determine the nullity of T? If so, find it.

(31) Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Is $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ in the span of $\mathcal{S}$? Is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ in the span of $\mathcal{S}$? What are the norms of the vectors in $\mathcal{S}$? What is the angle between them?

(32) Find bases for the range and null space of each of the following linear transformations. Also determine the rank and nullity of each transformation, and determine whether it is one-to-one or onto.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 2x_3 \end{bmatrix}$
(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 0 \\ 2x_1 - x_2 \end{bmatrix}$
(c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the rotation in $\mathbb{R}^3$ about the z-axis.

(33) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear,

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

What is $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$? Is T one-to-one?

(34) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{bmatrix}.$$

Find elementary matrices E and F such that $C = EAF$.

(35) Find the rank of the following matrices:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

(36) For each of the following matrices, compute the rank and inverse, if it exists.

(a) $\begin{bmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & -1 \end{bmatrix}$

(37) True or false: if the coefficient matrix of a system of m linear equations in n unknowns has rank m, then the system has a solution.

(38) Use Gaussian elimination to solve the following system of linear equations:

$$\begin{cases} 2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 = 1 \\ x_1 - x_2 + x_3 + 2x_4 - x_5 = 2 \\ 4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 = 6 \end{cases}$$

(39) What is the determinant of the projection map onto the xz-plane in $\mathbb{R}^3$?

(40) Verify that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$, and compute the $\mathcal{B}$-coordinate vectors of the standard basis vectors of $\mathbb{R}^2$.

(41) Suppose $T : \mathbb{R}^6 \to \mathbb{R}^6$ is a linear operator on $\mathbb{R}^6$ such that the range of T is contained in the null space of T. What is the largest possible rank the standard matrix of T could have?

(42) Let A be an $m \times n$ matrix, with $b \in \mathbb{R}^m$. Under what conditions is the solution set of the equation $Ax = b$ a subspace of $\mathbb{R}^m$?