PRACTICE PROBLEMS FOR THIRD MIDTERM

(1) State Rolle's Theorem, including all hypotheses.

(2) Verify that the function $f(x) = x^3 - 3x$ satisfies the hypotheses of the Mean Value Theorem on the interval $[-1, 1]$, and find all numbers c in this interval that satisfy the conclusion of the Mean Value Theorem.

(3) Show that if f is differentiable on an open interval I and $f'(x) < 1$ for all $x \in I$, then there is at most one point $c \in I$ such that $f(c) = c$.

(4) Find the largest intervals on which f increases and the largest intervals on which f decreases:
   (a) $f(x) = 2x - \dfrac{1}{x^2}$
   (b) $f(x) = |x + 1||x - 2|$
   (c) $f(x) = \cos 2x + 2\cos x$, $0 \le x \le 2\pi$.

(5) Find a function $f(x)$ such that $f'(x) = x^{1/3} - x^{1/2}$ for all $x \in \mathbb{R}$ and $f(0) = 1$.

(6) Sketch the graph of a differentiable function f that satisfies the given conditions, if possible. If it is not possible, explain why not.
   (a) $f(x) > 0$ for all x, $f(0) = 1$, and $f'(x) < 0$ for all x.
   (b) $f(1) = -1$, $f'(x) < 0$ for all $x \neq 1$, and $f'(1) = 0$.
   (c) $f(-1) = 4$, $f(2) = 2$, and $f'(x) > 0$ for all x.
   (d) $f(x) = 0$ only at $x = 1$ and at $x = 2$, $f(3) = 4$, $f(5) = -1$.
   (e) $f'(x) = 0$ for all $x \neq 0$, $f'(0) = 1$.
   (f) f has a local max at $x = -1$, a local min at $x = 1$, an absolute max of 6 at $x = 3$, and no absolute min.
   (g) f has a local max at $x = 4$, a local min at $x = 7$, an absolute min of 1 at $x = 0$, and no absolute max.

(7) Find all local extrema of the following functions:
   (a) $f(x) = x + \dfrac{1}{x}$
   (b) $f(x) = (1 - x)^2(1 + x)$
   (c) $f(x) = |x - 3| + |2x + 1|$
   (d) $f(x) = 2\sin^3 x - 3\sin x$, $0 < x < \pi$
   (e) $f(x) = |x^2 - 16|$

(8) Determine all local extrema of $f(x)$ if $f'(x) = x^3(x - 1)^2(x + 1)(x - 2)$.

(9) Find A, B, and C if $f(x) = Ax^2 + Bx + C$ has a local minimum at $x = 2$ and the graph of f passes through the points $(-1, 3)$ and $(3, -1)$.

(10) Find the absolute extreme values of:
   (a) $f(x) = \dfrac{x^2}{1 + x^2}$ on the interval $[-1, 2]$, and
   (b) $g(x) = \sin^4 x - \sin^2 x$ on the interval $[0, 2\pi/3]$.

(11) Give an example of a nonconstant function that takes on both its absolute max and absolute min on every open interval.

(12) Find a pair of numbers x and y such that $x + y = 16$ and $x^3 + y^3$ is as small as possible.

(13) Determine all inflection points and intervals of concavity of the following functions:
   (a) $f(x) = x^3 - 3x + 2$
   (b) $f(x) = 2\cos^2 x - x^2$
   (c) $f(x) = x\sqrt{4 - x^2}$

(14) Sketch the graphs of the following functions:
   (a) $f(x) = \sqrt{|x|}$
   (b) $f(x) = \dfrac{2x}{x^2 + 1}$
   (c) $f(x) = x^{1/3}(x - 6)^{2/3}$
   (d) $f(x) = \dfrac{x - 2}{x^2 - 5x + 6}$

(15) Find d given that $(d, f(d))$ is an inflection point of $f(x) = (x - a)(x - b)(x - c)$.

(16) Find all vertical and horizontal asymptotes of $f(x) = \dfrac{\sqrt{x}}{4\sqrt{x} - x}$.

(17) Determine whether $f(x) = x(x - 1)^{1/3}$ has a vertical tangent or a vertical cusp at $x = 1$.

(18) What is the maximum volume for a rectangular box (square base, no top) made from 12 square feet of cardboard?

(19) Two hallways, one 8 feet wide and the other 6 feet wide, meet at right angles. Determine the length of the longest ladder that can be carried horizontally from one hallway into the other. (You may express your answer in terms of a root of a suitable polynomial).

(20) Two boats are racing with constant speed toward a finish maker, boat A sailing from the south at 13 mph and boat B approaching from the east. When equidistant from the marker, the boats are 16 miles apart and the distance between them is decreasing at the rate of 17 mph. Which boat will win the race?

(21) The volume of a spherical balloon is increasing at the constant rate of 8 cubic feet per minute. How fast is the radius increasing when the diameter is exactly 20 feet? How fast is the surface area increasing at this time?