How to Show That a Limit Exists

Let $f(x)$ be a real-valued function whose domain includes an open interval containing $a \in \mathbb{R}$. In order to show that $\lim_{x \to a} f(x) = L$, it is necessary to show that:

For all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Hence the main step is "finding δ" (given ϵ). Usually δ will be some function of ϵ. Of course, as soon as an appropriate δ is found, any smaller value for δ will also work. A proof that $\lim_{x \to a} f(x) = L$ always has the following form: first write "Let $\epsilon > 0$ be arbitrary;" then determine what δ is (as a function of ϵ); then argue that the δ you have chosen actually works. Here are some examples of proofs that a given limit exists.

(1) $\lim_{x \to 4}(2x - 5) = 3$.

Proof. Let $\epsilon > 0$ be arbitrary. Set $\delta = \frac{1}{2}\epsilon$. Suppose $0 < |x - 4| < \delta$. Then $2|x - 4| < 2\delta = \epsilon$. Hence $2|x - 4| = |2x - 8| = |(2x - 5) - 3| < \epsilon$. □

(2) $\lim_{x \to 1}(x^2 - 1) = 0$.

Proof. Let $\epsilon > 0$ be arbitrary. Set $\delta = \min\{\frac{\epsilon}{3}, 1\}$, so that $\delta \leq 1$. Suppose $0 < |x - 1| < \delta$. Then $0 < x < 2$, so $x + 1 = |x + 1| < 3$. Hence $|x^2 - 1| = |x + 1|\,|x - 1| < 3\frac{\epsilon}{3} = \epsilon$. □

(3) $\lim_{x \to 1} x^3 = 1$.

Proof. Let $\epsilon > 0$ be arbitrary. Set $\delta = \min\{\frac{\epsilon}{7}, 1\}$, so that $\delta \leq 1$. Suppose $0 < |x - 1| < \delta$. Then $0 < x < 2$, so $x^2 + x + 1 = |x^2 + x + 1| < 7$. Hence $|x^3 - 1| = |x^2 + x + 1|\,|x - 1| < 7\frac{\epsilon}{7} = \epsilon$. □

(4) $\lim_{x \to 3} \sqrt{x + 1} = 2$.

Proof. Let $\epsilon > 0$ be arbitrary. Set $\delta = \min\{\epsilon, 1\}$, so that $\delta \leq 1$. Suppose $0 < |x - 3| < \delta$. Then $2 < x < 4$, so $\sqrt{3} + 2 < |\sqrt{x + 1} + 2| < \sqrt{5} + 2$. In particular, $1 < |\sqrt{x + 1} + 2|$. Since $|x - 3| = |\sqrt{x + 1} - 2|\,|\sqrt{x + 1} + 2| < \delta$, it follows that $|\sqrt{x + 1} - 2| < \dfrac{\delta}{|\sqrt{x + 1} + 2|} < \delta \leq \epsilon$. □

Limit Definitions

Here are the fifteen different notions of limit that we have defined so far. I have written the definitions using logical shorthand for brevity, but you should be able to rewrite them in English if necessary. You will be responsible for knowing each of these definitions! However, it should not be necessary to memorize each one separately. Try to understand what the notion of a limit *means*. If you succeed in doing this, then you should be able to give the definition of any of the following as they come up, without having to memorize anything.

(1)	$\lim_{x\to a} f(x) = L$	$\Leftrightarrow$	$\forall \epsilon > 0, \exists \delta > 0$ st $0 < \lvert x-a\rvert < \delta \Rightarrow \lvert f(x) - L\rvert < \epsilon$
(2)	$\lim_{x\to a^+} f(x) = L$	$\Leftrightarrow$	$\forall \epsilon > 0, \exists \delta > 0$ st $0 < x-a < \delta \Rightarrow \lvert f(x) - L\rvert < \epsilon$
(3)	$\lim_{x\to a^-} f(x) = L$	$\Leftrightarrow$	$\forall \epsilon > 0, \exists \delta > 0$ st $0 < a-x < \delta \Rightarrow \lvert f(x) - L\rvert < \epsilon$
(4)	$\lim_{x\to a} f(x) = +\infty$	$\Leftrightarrow$	$\forall N \in \mathbb{N}, \exists \delta > 0$ st $0 < \lvert x-a\rvert < \delta \Rightarrow f(x) > N$
(5)	$\lim_{x\to a^+} f(x) = +\infty$	$\Leftrightarrow$	$\forall N \in \mathbb{N}, \exists \delta > 0$ st $0 < x-a < \delta \Rightarrow f(x) > N$
(6)	$\lim_{x\to a^-} f(x) = +\infty$	$\Leftrightarrow$	$\forall N \in \mathbb{N}, \exists \delta > 0$ st $0 < a-x < \delta \Rightarrow f(x) > N$
(7)	$\lim_{x\to a} f(x) = -\infty$	$\Leftrightarrow$	$\forall N \in \mathbb{N}, \exists \delta > 0$ st $0 < \lvert x-a\rvert < \delta \Rightarrow f(x) < -N$
(8)	$\lim_{x\to a^+} f(x) = -\infty$	$\Leftrightarrow$	$\forall N \in \mathbb{N}, \exists \delta > 0$ st $0 < x-a < \delta \Rightarrow f(x) < -N$
(9)	$\lim_{x\to a^-} f(x) = -\infty$	$\Leftrightarrow$	$\forall N \in \mathbb{N}, \exists \delta > 0$ st $0 < a-x < \delta \Rightarrow f(x) < -N$
(10)	$\lim_{x\to +\infty} f(x) = L$	$\Leftrightarrow$	$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $x > N \Rightarrow \lvert f(x) - L\rvert < \epsilon$
(11)	$\lim_{x\to -\infty} f(x) = L$	$\Leftrightarrow$	$\forall \epsilon > 0, \exists N \in \mathbb{N}$ st $x < -N \Rightarrow \lvert f(x) - L\rvert < \epsilon$
(12)	$\lim_{x\to +\infty} f(x) = +\infty$	$\Leftrightarrow$	$\forall N \in \mathbb{N}, \exists M \in \mathbb{N}$ st $x > M \Rightarrow f(x) > N$
(13)	$\lim_{x\to -\infty} f(x) = +\infty$	$\Leftrightarrow$	$\forall N \in \mathbb{N}, \exists M \in \mathbb{N}$ st $x < -M \Rightarrow f(x) > N$
(14)	$\lim_{x\to +\infty} f(x) = -\infty$	$\Leftrightarrow$	$\forall N \in \mathbb{N}, \exists M \in \mathbb{N}$ st $x > M \Rightarrow f(x) < -N$
(15)	$\lim_{x\to -\infty} f(x) = -\infty$	$\Leftrightarrow$	$\forall N \in \mathbb{N}, \exists M \in \mathbb{N}$ st $x < -M \Rightarrow f(x) < -N$