

THE PRECISE DEFINITION OF A LIMIT

Suppose that $y = f(x)$ is a real-valued function whose domain includes an open interval containing the point $a \in \mathbb{R}$. Then we say that $L \in \mathbb{R}$ is the *limit* of $f(x)$ as x approaches a , and write

$$\lim_{x \rightarrow a} f(x) = L,$$

if the following holds:

- for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Sometimes we make use of the following logical shorthand when talking about limits:

- (1) “for all” — \forall
- (2) “there exists” — \exists
- (3) “such that” — st
- (4) “implies” — \Rightarrow

With this shorthand, we may write:

$$\lim_{x \rightarrow a} f(x) = L \quad \text{iff} \quad \forall \epsilon > 0 \exists \delta > 0 \text{ st } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

Negating this definition, we have that $\lim_{x \rightarrow a} f(x)$ does *not* equal L iff:

$$\exists \epsilon > 0 \text{ st } \forall \delta > 0 \ |f(x) - L| \geq \epsilon \text{ for some } x \text{ satisfying } 0 < |x - a| < \delta.$$

In particular, $\lim_{x \rightarrow a} f(x)$ *does not exist* if the above holds for all $L \in \mathbb{R}$; ie, $\lim_{x \rightarrow a} f(x)$ does not exist iff:

$$\forall L \in \mathbb{R} \exists \epsilon > 0 \text{ st } \forall \delta > 0 \ |f(x) - L| \geq \epsilon \text{ for some } x \text{ satisfying } 0 < |x - a| < \delta.$$

Notice that $\lim_{x \rightarrow a} f(x)$ *does not depend in any way* on the value $f(a)$, or even on the existence of $f(a)$.

We may replace the ‘ a ’ or the ‘ L ’ in “ $\lim_{x \rightarrow a} = L$ ” to obtain variations on the notion of limit. Specifically, we may replace $L \in \mathbb{R}$ with $+\infty$ or $-\infty$ to obtain the notion of $f(x)$ becoming *unbounded*; we may replace a with $+\infty$ or $-\infty$ to obtain *limits at infinity*; and we may replace a with a^- or a^+ to obtain *one-sided limits*.

Over the weekend, you should familiarize yourself with ϵ - δ definitions by attempting to write the precise definition of these variations on the notion of a limit. In particular, attempt to write out precise definitions of the following for Monday:

$$\lim_{x \rightarrow a^-} f(x) = +\infty, \quad \lim_{x \rightarrow +\infty} f(x) = L, \quad \lim_{x \rightarrow a^+} = -\infty.$$