TECHNIQUES FOR GRAPHING FUNCTIONS

Given a function $y = f(x)$, our goal is to sketch a rough graph of f that reflects important features such as:

- intercepts
- asymptotes and end behavior
- local extrema
- intervals on which f increases and decreases
- inflection points
- intervals of concavity
- points of discontinuity or non-differentiability, including cusps and vertical tangents

Many of the techniques of calculus developed in Chapter 4 can help us do this. What follows is a general summary of graphing techniques, which can also be used as a checklist when trying to graph a particular given function.

(1) Be able to graph (in some cases, only roughly) the following basic functions from memory:
  - lines (including y-intercept, point-slope, and general linear forms)
  - parabolas (vertex and general linear form, factoring, completing the square, determining orientation and degree of "narrowness / wideness")
  - general shape of polynomials, given the degree and leading coefficients (end behavior, how many "humps," behavior near a root depending on its degree, etc — in particular, you should be able to graph fairly accurately a polynomial that has been fully factored)
  - the power functions $y = x^a$, $a > 0$ (and especially a an integer)
  - the reciprocal function $y = x^{-1}$,
  - the absolute value function, and the function $y = |x|/x$
  - the six trig functions, but especially $y = \sin x$, $y = \cos x$, and $y = \tan x$

(2) Translations, dilations and compressions. Specifically, given the graph of $y = f(x)$, know how to obtain from it the graph of:
  - $f(x) + c$, $c > 0$ (vertical shift up)
  - $f(x) - c$, $c > 0$ (vertical shift down)
  - $f(x + c)$, $c > 0$ (horizontal shift left)
  - $f(x - c)$, $c > 0$ (horizontal shift right)
  - $cf(x)$, $c > 1$ (vertical stretch)
  - $f(cx)$, $c > 1$ (horizontal compression)
  - $cf(x)$, $0 < c < 1$ (vertical compression)
  - $f(cx)$, $0 < c < 1$ (horizontal stretch)
  - $-f(x)$ (reflection over x-axis)
  - $f(-x)$ (reflection over y-axis)

(3) Identify the domain and range of the function you are trying to graph.
(4) Find the intercepts of the function you are trying to graph.
  - to find the y-intercept, just plug in $x = 0$
  - to find the x-intercept(s), solve $f(x) = 0$
(5) If possible, use symmetry or periodicity to help you graph the function.
  - if $f(x) = f(-x)$ for all x, then f is an even function
  - if $f(x) = -f(-x)$ for all x, then f is an odd function
  - the trig functions are periodic with period 2π
(6) Find asymptotes and end behavior of the function you are trying to graph.
  - f has a horizontal asymptote at $y = L$ if $\lim_{x\to\pm\infty} f(x) = L$
  - f has a vertical asymptote at $x = a$ if $\lim_{x\to a^\pm} f(x) = \pm\infty$ (these usually occur where a denominator is zero)
  - remember that all rational functions exhibit the same end behavior as a suitable monomial
(7) Find the intervals on which the function increases and decreases
  - a differentiable function f increases on an open interval I if $f' > 0$ on I
  - a differentiable function f decreases on an open interval I if $f' < 0$ on I
(8) Find the local extrema of the function
  - if f is differentiable, use the first or second derivative tests
  - remember that local extrema always occur at critical points
(9) Find the intervals of concavity of the function
  - a twice-differentiable function f is concave up on an open interval I if $f'' > 0$ on I
  - a twice-differentiable function f is concave down on an open interval I if $f'' < 0$ on I
(10) Find the inflection points of the function
  - test each of the critical points of f'' for a sign change in f''
(11) If all else fails, construct a table of values and plot points!

For practice, try graphing the following functions, making sure to label intercepts, asymptotes, local extrema, inflection points, cusps, etc.

(1) $y = x^3 + 6x^2 + 9x$

(2) $y = \dfrac{x-1}{x^2}$

(3) $y = \dfrac{x}{\sqrt{x^2+1}}$

(4) $y = x^{5/3} - 5x^{2/3}$

(5) $y = \sin x - x$

(6) $y = x\sqrt{5-x}$