PRACTICE PROBLEMS FOR FINAL EXAM

(1) Find all local extrema of the function $g(x) = \int_0^x (t+1)(t-1)\,dt$.

(2) Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} 3\left(1+\frac{2k}{n}\right)^2 \left(\frac{2}{n}\right)$ by interpreting it as a limit of Riemann sums.

(3) State the Intermediate Value Theorem, including all hypotheses, and give an example of a function $y = f(x)$ and a closed interval $[a, b]$ such that $f(a) < 0 < f(b)$, but $f(x) \neq 0$ for all $x \in (a, b)$.

(4) Implicitly differentiate the equation $\sqrt{xy} = 1 + x^2 y$ with respect to x, and then again with respect to y.

(5) Find the absolute extrema of the function $f(x) = \frac{x}{x^2+1}$ on the interval $[0, 2]$.

(6) On what interval(s) is the function $f(x) = \int_0^x (t-a)(t-b)\,dt$ concave up?

(7) Compute the Riemann sum for $f(x) = \sqrt{x}$ on the interval $[1, 7]$, using a regular partition with $n = 3$ subintervals, and midpoints as sample points.

(8) Give the precise definition of the statement $\lim_{x\to a^+} f(x) = +\infty$. Then give an example of a function f and number $a \in \mathbb{R}$ such that $\lim_{x\to a^+} f(x) = +\infty$.

(9) Find the area of the region inside the parabola $y = -x^2 + 5x - 4$ and above the x-axis.

(10) Find the area of the region between $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \pi/4$.

(11) Give an example of everywhere continuous functions f and g such that $f + g$ is differentiable everywhere but f fails to be differentiable at some point.

(12)
- Define $g(x) = \int_0^x (t-1)(t-2)\,dt$ for all $x \geq 0$. Find the intervals on which $g(x)$ increases, and those on which $g(x)$ decreases.
- More generally, define $g(x) = \int_0^x f(t)\,dt$ for all $x \geq 0$, for $y = f(t)$ some fixed continuous function of t. How can you determine the intervals on which g increases and decreases?

(13) Find the largest number $\delta > 0$ such that $|x^2 - 9| < \frac{1}{10}$ whenever $0 < |x - 3| < \delta$.

(14) Find the largest number $\delta > 0$ such that $\frac{1}{x^2} > 10^{1000}$ whenever $0 < |x| < \delta$.

(15) Find local extrema, inflection points, intervals of increase and decrease, and intervals of concavity of the functions $f(x) = 3x^{2/3} - x$ and $g(x) = x\sqrt{x+3}$.

(16) State the definition of the derivative and use it to compute $\frac{d}{dx}(x^2 + 2x + 1)$.

(17) Express the definite integral $\int_0^1 x^3\,dx$ as a limit of Riemann sums using regular partitions and left endpoints as sample points. Evaluate this limit given that $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$, and check your answer using the FTOC(2).

(18)
- Define $g(h) = \frac{1}{h}\int_0^h \sqrt{t}\,dt$ for all $h \geq 0$. Find the intervals on which $g(h)$ increases, and those on which $g(h)$ decreases.
- More generally, define $g(h) = \frac{1}{h}\int_0^h f(t)\,dt$ for all $h \geq 0$, for $y = f(t)$ some fixed continuous function of t. How can you determine the intervals on which g increases and decreases?

(19) Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \left[\sec^2\left(\frac{k\pi}{4n}\right)\tan^3\left(\frac{k\pi}{4n}\right)\frac{\pi}{4n}\right]$ and $\lim_{n\to\infty} \sum_{k=1}^{n} \left[2\frac{k\sqrt{\pi/3}}{n}\cos\left(\frac{k^2\pi}{3n^2}\right)\frac{\sqrt{\pi/3}}{n}\right]$.

(20) State the Fundamental Theorem of Calculus (Part 1), including all hypotheses.

(21) Find all asymptotes, cusps, and vertical tangents of the functions $f(x) = \sqrt{x^2+1} - x$ and $g(x) = x\tan x$.

(22) Find a function $g(x)$ such that $g'(x) = x^2 \sin x$ and $g(\pi) = 0$.

(23) Prove that $\lim_{x\to 0} |x| = 0$, $\lim_{x\to 2}(3x-1) = 5$, and $\lim_{x\to 0}\left(x\,\sin\frac{1}{x}\right) = 0$

(24) Suppose $f(x) = ax + b$ with $a > 0$, and fix a closed interval $[c, d]$ and a partition $\mathcal{P}$ of $[c, d]$. Let R_{left}, R_{right}, R_{min}, R_{max} and R_{mid} denote the Riemann sums for f on $[c, d]$ relative to $\mathcal{P}$ using, respectively, left endpoints, right endpoints, minima, maxima, and midpoints as sample points. Order R_{left}, R_{right}, R_{min}, R_{max} and R_{mid} from least to greatest, indicating equality where appropriate. Are any of these Riemann sums equal to $\int_c^d f(x)\,dx$? Explain.

(25) At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

(26) Evaluate $\int_0^{\pi/4} 2\sin x \cos x\,dx$.

(27) Sketch the graph of the function $y = \frac{x}{(x-1)^2}$, making sure to label any intercepts, asymptotes, local extrema, and inflection points; also state the intervals on which y is increasing, decreasing, concave up, and concave down. Note: $y' = \frac{x+1}{-(x-1)^3}$, and $y'' = \frac{2(x+2)}{(x-1)^4}$.

(28) Find an equation of the line tangent to the curve $y = (x+1)^2 + \sin x + \int_0^x \sec\theta \, d\theta$ at $x = 0$.

(29) State Fermat's Theorem, including all hypotheses, and give an example to show that its converse is false.

(30) A kite 100 feet above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string have been let out?

(31) Define $f_0(x) = |x|$, and for each positive integer $n \geq 1$, define $f_n(x) = |f_{n-1}(x) - 1|$. What are the critical points of $f_{99}(x)$?

(32) Find $g'(x)$ if $g(x) = \int_0^{\sin^2 x^2} 3\sec\theta \frac{\theta}{1+\tan\sqrt{\theta}} \, d\theta$.

(33) Find $\lim_{h\to 0} \frac{1}{h} \int_{\pi/2}^{(\pi/2)+h} x \sin x \, dx$.

(34) Evaluate the following limits
- $\lim_{x\to 0} \frac{\cot x}{\csc x}$
- $\lim_{x\to 0} x \cot x$
- $\lim_{t\to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right)$

(35) Differentiate the following functions:
- $y = \frac{x^3 - 2x\sqrt{x}}{x}$
- $y = \sin(\tan\sqrt{\sin x})$
- $y = (2x-5)^4(8x^2-5)^{-3}$

(36) Evaluate the following integrals:
- $\int_1^4 \sqrt{t}(1+t) \, dt$
- $\int_1^2 \frac{y + 5y^7}{y^3} \, dy$
- $\int \sec\theta \tan\theta \, d\theta$

PROPERTIES OF THE DEFINITE INTEGRAL

Suppose f and g are continuous on $[a,b]$ and let $c \in \mathbb{R}$. Then the following hold:

(1) if $a = b$, then $\int_a^b f(x)\,dx = 0$

(2) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(3) $\int_a^b c\,dx = c(b-a)$

(4) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(5) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(6) $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

(7) if $a \le d \le b$, then $\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx$

(8) if $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int g(x)\,dx$

(9) if $m \le f(x) \le M$ for all $x \in [a,b]$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

(10) if f is an odd function, then $\int_{-c}^c f(x)\,dx = 0$, and if f is an even function, then $\int_{-c}^c f(x)\,dx = 2\int_0^c f(x)\,dx$