# HOW TO SHOW THAT A LIMIT DOES NOT EXIST

Let $f(x)$ be a real-valued function whose domain includes an open interval containing $a \in \mathbb{R}$. In order to show that $\lim_{x\to a} f(x)$ does not exist, it is necessary to show that for all real numbers $L \in \mathbb{R}$, $\lim_{x\to a} f(x) \neq L$. Thus it is necessary to prove:

> For all $L \in \mathbb{R}$, there exists $\epsilon > 0$ such that for all $\delta > 0$, $|f(x) - L| \geq \epsilon$ for some $x \in \mathbb{R}$ satisfying $0 < |x - a| < \delta$.

It is frequently necessary to split up such a proof into different cases for different values of the purported limit $L$. Notice also that as soon as you have "found an $\epsilon$" when showing that a given limit does not equal a particular value $L$, any *smaller* $\epsilon$ will also work. Here are some examples of proofs that a given limit does not exist.

(1) $\lim_{x\to 0} \frac{1}{x^2}$ does not exist.

*Proof.* Let $L \in \mathbb{R}$ be arbitrary. Set $\epsilon = 1$, and let $\delta > 0$ be arbitrary. Then there exists $x \in \mathbb{R}$ such that $0 < x < \delta$ and $L + 1 < \frac{1}{x^2}$. Hence $\lim_{x\to 0} \frac{1}{x^2} \neq L$. Since $L$ was arbitrary, this shows that $\lim_{x\to 0} \frac{1}{x^2}$ does not exist. □

(2) $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

*Proof.* Let $L \in \mathbb{R}$ be arbitrary, set $\epsilon = 1$, and let $\delta > 0$ be arbitrary. We consider two cases: $L \leq 0$ and $L > 0$. First suppose $L \leq 0$. Then for any $x \in \mathbb{R}$ such that $0 < x < \delta$, we will have $\frac{|x|}{x} = 1$, so that in particular $\left|\frac{|x|}{x} - L\right| \geq 1 = \epsilon$. Hence $L$ cannot be the limit. Next suppose that $L > 0$. Then for any $x \in \mathbb{R}$ such that $-\delta < x < 0$, we will have $\frac{|x|}{x} = -1$, so that in particular $\left|\frac{|x|}{x} - L\right| \geq 1 = \epsilon$. Again, we conclude that $L$ cannot be the limit, and therefore, the limit $\lim_{x\to 0} \frac{|x|}{x}$ does not exist. □

(3) $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

*Proof.* Let $L \in \mathbb{R}$ be arbitrary, set $\epsilon = 1$, and let $\delta > 0$ be arbitrary. We consider two cases: $L \leq 0$ and $L > 0$. First suppose $L \leq 0$. Fix $n \in \mathbb{N}$ large enough so that $\frac{1}{\pi/2 + 2n\pi} < \delta$, and set $x = \frac{1}{\pi/2 + 2n\pi}$. Notice that $\sin \frac{1}{x} = 1$. Then $0 < x < \delta$, but $\left|\sin \frac{1}{x} - L\right| \geq 1 = \epsilon$, so $L$ cannot be the limit. Now suppose $L > 0$. Fix $n \in \mathbb{N}$ large enough so that $\frac{1}{3\pi/2 + 2n\pi} < \delta$, and set $x = \frac{1}{3\pi/2 + 2n\pi}$. Notice that $\sin \frac{1}{x} = -1$. Then $0 < x < \delta$, but $\left|\sin \frac{1}{x} - L\right| \geq 1 = \epsilon$, so again $L$ cannot be the limit. We conclude that $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist. ☐

(4) Let $f$ be the Dirichlet function defined by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$, and let $c \in \mathbb{R}$. Then $\lim_{x \to c} f(x)$ does not exist.

*Proof.* Let $L \in \mathbb{R}$ be arbitrary, set $\epsilon = \frac{1}{2}$, and let $\delta > 0$ be arbitrary. We consider two cases: $L \leq \frac{1}{2}$ and $L > \frac{1}{2}$. First suppose $L \leq \frac{1}{2}$. Then there is a rational number $x \in \mathbb{Q}$ such that $0 < |x - c| < \delta$, but $f(x) = 1$ and so $|f(x) - L| \geq \frac{1}{2} = \epsilon$. Hence $L$ cannot be the limit. Now suppose $L > \frac{1}{2}$. Then there is an irrational number $x \in \mathbb{R}$ such that $0 < |x - c| < \delta$, but $f(x) = 0$ and so $|f(x) - L| \geq \frac{1}{2} = \epsilon$. Hence again $L$ cannot be the limit, and so we conclude that $\lim_{x \to c} f(x)$ does not exist. ☐

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In this class, you will be asked on homework, workshops, and exams to determine whether or not a given limit exists. You may be asked in homework and workshops to prove that a limit does not exist, and you may be asked on exams to prove that a given limit *does* exist. However, you will NOT be asked on exams to write a complete proof (such as those given above) that a given limit does not exist.

Still, you should try to understand the arguments that are given in order to show that a particular limit does not exist. Try to focus on the ideas behind the argument and not worry so much about "how to write it." Remember that the key step in proving that a limit does not exist is *finding* $\epsilon$. In the four examples given above, you should go back and determine whether or not the largest possible value for $\epsilon$ was given, and if not, what this largest value is.