MATH 642:581–Spring, 2014
Assignment 5–Due April 23 (April 21 version)

A few preliminary remarks.

1. Follow the general instructions for homework given in:

   http://www.math.rutgers.edu/∼saks/homework-grad.html

2. Please be on the look out for errors. If something seems not to make sense, check with me before investing a lot of time on the problem. I would appreciate being notified of any typos (even minor ones).

PROBLEMS

1. (a) Prove that for any graph G, $\chi(G)\chi(\bar{G}) \geq |V(G)|$.
   (b) Prove that for any graph G, $\chi(G) + \chi(\bar{G}) \geq 2\sqrt{|V(G)|}$.
   (c) Show that for each perfect square n there is a graph on n vertices for which the previous two bounds are tight.
   (d) Prove that $\chi(G) + \chi(\bar{G}) \leq |V(G)| + 1$.

2. A graph is *outerplanar* if it can be drawn in the plane so that every vertex is in the boundary of the outer face. A graph is *maximal outerplanar* if it is outerplanar and any graph obtained by adding an edge (to the existing vertex set) is not outerplanar.

   (a) State and prove a theorem that expresses the number of edges of a maximal outerplanar graph in terms of the number of vertices.
   (b) Prove that K_4 is not outerplanar.
   (c) Prove that $K_{2,3}$ is not outerplanar.
   (d) Prove that if G is not outerplanar, then G contains a TK_4 or a $TK_{2,3}$. (Hint provided below.)

3. The proof that every planar graph is 5 colorable (Theorem 8 in Bollobás; to be covered in class) can be modified to the following (false) proof that every planar graph is 4 colorable.

   As in the proof of the 5 color theorem, let G be a counterexample with the fewest number of vertices and consider a planar embedding of G. We may assume that G is maximal planar. Let v be a vertex of degree at most 5 in G and let c be a 4-coloring of $G - v$. If the neighbors of v are colored by c with less than 3 colors then c can be extended to a 4-coloring of G so c uses all four colors on $N(v)$. So $deg(v) \geq 4$. If v has degree 4 then the neignbors of v induce a cycle C with vertices x_1, x_2, x_3, x_4 in order and x_i colored by color i. As in the proof of the 5CT there must be a path consisting of vertices colored 1 or 3 from x_1 to x_3 and a path consisting of vertices colore 2 or 4 from x_2 to x_4 and these paths lie in the outer face of C which means they must have a vertex in common, a contradiction. Suppose v has degree 5, so the neighbors of v induce a cycle C with vertices x_1, x_2, x_3, x_4, x_5 in order. Since all 4 colors appear on C, we may assume wlog that x_i is colored i for $1 \leq i \leq 4$ and x_5 is colored 2.

Then there is a color 1,3 path linking x_1 to x_3 outside of C which means there is no color 2,4 path linking x_2 to x_4. Similarly there is a color 1,4 path linking x_1 to x_4 outside of C, but then there is no color $2,3$ path linking x_5 to x_3. So recolor the $2,4$ component of x_2 by interchanging colors 2 and 4, and recolor the 2,3 component containing x_5. Then we have a new coloring in which x_2 is color 4 and x_5 is color 3 and color 2 does not appear on the cycle C. So we can extend the coloring to a coloring of G by coloring v by 2.

What is the falacy in this proof? Find a concrete example of a plane graph where the above "recipe" for coloring it fails.

4. (Bollabás, problem 5.36.) Prove that if the graph G has an orientation having no directed path with more than k vertices, then $\chi(G) \leq k$. (A hint is given in Bollobás.)

5. Let G be a triangle free graph.

   (a) Prove: $\chi(G) \leq \sqrt{2|V(G)|}$. (Hint provided below.)

   (b) For any graph H and positive integer d if H has a subgraph of minimum degree at least d then $\chi(H) \leq \max(d, \chi(H'))$ where H' is any subgraph of H that is maximal among subgraphs of H of minimum degree at least d. (Note that if H itself has minimum degree at least d then this is a triviality.)

   (c) Prove: $\chi(G) \leq \lceil (4|E(G)|)^{1/3} \rceil$.

6. Consider the following graph coloring game between two players Builder (who builds a graph) and Colorer (who colors the graph). The graph starts out empty. The game proceeds in a sequence of n rounds. At round j Builder adds a new vertex v_j to the graph and adds edges from v_j to some subset of previous vertices (of his choice). Colorer then assigns a color to vertex v_j, always maintaining that the graph is properly colored. (Vertices colored in previous rounds may not be recolored.) Let C be the number of colors used by Colorer throughout the game and let χ be the chromatic number of the resulting graph. Colorer seeks to minimize his *regret* which is the ratio C/χ (which is between 1 and n). The goal of this problem is to show that Builder has a strategy that forces the regret of Colorer to be at least $\Omega(n/(log_2 n)^2)$. (This is fairly amazing, you might want to think about this before reading further.)

   Here is the strategy of the Builder. Let k be the least integer such that $k2^{k-1} \geq n$. As the game proceeds, to guide his strategy, Builder constructs a table with k columns and with rows labeled by the colors used by Colorer so far. (So every time a new color is used by Colorer, a row is added to the table.) In round j Builder will use the table to decide which vertices v_j should be joined to. After Colorer colors vertex v_j, Builder will place vertex v_j into the table. Builder will maintain the following properties for the table: (A) For each color r that has been used by Colorer, the vertices colored by r are in distinct locations of row r (B) The set of vertices in each column is independent.

   Let T_{j-1} be the table constructed up to the begining of round j, so there are $j-1$ vertices in the table. The *support* of color r in T_{j-1} is the set $S(r)$ of columns for which the entry in row r is filled. Builder proceeds as follows: If there is some nonempty subset I of columns which is not the support of any color, Builder selects such a subset I and connects v_j to all of the vertices that are NOT in the columns indexed by I.

(a) Show that after Colorer colors v_j, it is possible for Builder to place v_j in the table so as to maintain (A) and (B).

(b) Show that for every step $j \leq n$, it is possible for Builder to carry out the above strategy.

(c) Prove that if n is sufficiently large then the regret of Colorer is at least $n/(\log_2(n))^2$.

Some hints

Problem 2 For the last part, consider a graph that is not outerplanar, and consider a subgraph with the minimum number of edges that is not outerplanar. Show that (after removing any isolated vertices) this graph is 2-connected, and planar. Consider a planar embedding of the graph.

Problem 5 . For part (a) prove by induction on k that if k is a positive integer and $|V(G)| < (k+1)^2/2$ then $\chi(G) \leq k$. Divide into cases according to the maximum degree of G. For part (c), use part (b) with part (a) and a careful choice of d.