MATH 642:581–Spring, 2014
Assignment 4–Due April 9 (April 7 version)

A few preliminary remarks.

1. Follow the general instructions for homework given in:

   http://www.math.rutgers.edu/∼saks/homework-grad.html

2. Please be on the look out for errors. If something seems not to make sense, check with me before investing a lot of time on the problem. I would appreciate being notified of any typos (even minor ones).

PROBLEMS

1. Let G be a graph. Let F_G be the set of nonnegative integer valued functions f on the vertex set such that $\sum_v f(v) = |V(G)|$. For a function $f \in F_G$ let $s(f) = s_G(f)$ be the sum over all edges vw of $f(v)f(w)$.

   (a) Prove that among all functions in F_G, $s(f)$ is maximized when the support of f is a clique of G. (Hint provided below.)

   (b) Determine the maximum of $s_G(f)$ in terms of some simple parameters associated with G.

   (c) Use the previous result to obtain an alternative proof that for any $n \geq r \geq 1$, any graph on n vertices with more than $t_{r-1}(n)$ edges has a K_r subgraph.

2. (Bollobas, problem 4.3) Let $0 \leq k \leq n$. Show that an n by n bipartite graph without $k+1$ independent edges has at most kn edges. Determine (with proof) the unique extremal graph.

3. (Bollobas, problem 4.10, with steps) The purpose of this problem is to show that any graph with minimum degree at least 3 has a TK_4 subgraph. We'll show something stronger: Any graph with at least 4 vertices that has no TK_4 subgraph has two non-adjacent vertices of degree at most 2. We'll proceed by induction on the number of vertices; the base case of 4 vertices is trivial. Let G be a graph with no TK_4 subgraph. We'll show that G has 2 non-adjacent vertices

   (a) Show that G can't be 3-connected. (Hint provided below.)

   (b) Finish the argument in the case that G is disconnected.

   (c) Finish the argument in the case that G is connected and has at least one cut point.

   (d) Finish the argument in the case that G has a cut set of size 2, but no cut set of size 1. (Hint provided below.)

   (Hint provided below.)

4. (Bollobas Problem 22, corrected and paraphrased.)

   (a) Show that for any n-vertex graph, there is a set of at most $\lfloor n^2/4 \rfloor$ edges and triangles that cover all of the edges (meaning that every edge of the graph either belongs to the set, or is in one of the triangles that belongs to the set).

(b) If $\mathcal{F}$ is a collection of sets, the intersection graph of $\mathcal{F}$ is the graph whose vertex set is $\mathcal{F}$ with edges F_1F_2 if $F_1 \cap F_2 \neq \emptyset$. Prove that for any n vertex graph G there is a collection of sets $\mathcal{F}$ whose intersection graph is isomorphic to G and whose union has size at most $\lfloor n^2/4 \rfloor$.

5. The purpose of this problem is to prove the following: For any integer $r \geq 1$ there is a constant $c(r)$ such that $ext(n, K_{r,r}) \leq c(r)n^{2-1/r}$. (Hints are available below.)

   (a) For $0 < s \leq t < n$, let $z(n, s, t)$ be the maximum number of edges in a bipartite graph with both parts of size n that has no $K_{s,t}$. Prove that $ex(n, K_{s,t})$ is between $z(\lfloor n/2 \rfloor, s, t)$ and $z(n, s, t)/2$.

   (b) Let G be a bipartite graph with bipartition (A, B) where $|A| = |B| = n$, that has no $K_{r,r}$. Prove that: $\sum_{x \in A} \binom{d(x)}{r} \leq (r-1)\binom{n}{r}$.

   (c) Deduce that there is a constant $c(r)$ such that for all n $ext(n, K_{r,r}) \leq c(r)n^{2-1/r}$.

6. Let $\mathcal{V}$ be a near-partition of G with at least k parts. For disjoint subsets A, B of V, let $\rho(A, B) = |E(A, B)|/|A||B|$.

   Given any subset W of $V(G)$, define $C_{\mathcal{V}}(W)$ to be the sum over all pairs A, B of distinct parts of $\mathcal{V}$ of $\rho(A, B)|A \cap W||B \cap W|$.

   Prove that for any $\gamma > 0$, there are constants $K(\gamma)$ and $\varepsilon(\gamma) > 0$ such that if $\mathcal{V}$ is an $\epsilon(\gamma)$-regular partition of G with at least $K(\gamma)$ parts then $C_{\mathcal{V}}(W)$ gives an estimate for $|E(G[W])|$ that is within $\gamma|V(G)|^2$.

Some hints

Problem 1(a) Suppose $g \in F_G$ is a function that maximizes $s_G(f)$ having smallest possible support and show by contradiction that $supp(g)$ is a clique.

Problem 3 For part a, consider a cycle of smallest size in G. For part d, let $\{x, y\}$ be the cut set, and let $V_1, \ldots, V_k$ be the connected components of $G - \{x, y\}$ and let G_i be the graph obtained from $G[V_i \cup \{x, y\}]$ by adding the edge xy (it it's not already present). Show that G_i has no TK_4.

Problem 5 Part b. Count the number of pairs (a, Y) where $a \in A$, $Y \subseteq B$, $|Y| = r$ and $Y \subseteq N(a)$. Part c. Use the bounds $(s/t)^t \leq \binom{s}{t} \leq s^t$.