MATH 642:581–Spring, 2014

Assignment 3–Due March 12 (March 10 version)

A few preliminary remarks.

1. Follow the general instructions for homework given in:

   http://www.math.rutgers.edu/$\sim$saks/homework-grad.html

2. Please be on the look out for errors. If something seems not to make sense, check with me before investing a lot of time on the problem. I would appreciate being notified of any typos (even minor ones).

3. The problems in this homework make use of the following theorems: Bollobas Theorem 3.7 and Corollary 3.9 (this is the König-Hall-Egervary theorem for bipartite graphs) and Theorem Corollary 15 (Tutte's theorem.) The first theorem was presented at the end of class on February 19. We will discuss those theorems during class on March 4 and 5. The problems below make use of the theorem statements and not the proofs.

PROBLEMS

1. (Bollobas, problem 3.26) Let G be a graph on $2k+2$ vertices that has a 1-factor, and also has the property that every matching of size k can be extended to a 1-factor. Prove that every matching of size $k-1$ can be extended to a 1-factor.

2. (Bollobas, problem 32.) Prove that a 2-edge-connected cubic graph has a 1-factor. (A hint is given in Bollobas.) Show also that a cubic graph need not have a 1-factor.

3. (Bollobas, problem 3.38, paraphrased.) A graph G is ν-maximal if any graph on the same vertex set obtained by adding an edge has largest matching larger than the largest matching in G. State and prove a theorem of the following form: For any positive integers n, k with $n \geq 2k$, the ν-maximal graphs with n vertices and $\nu(G) = k$ have the following structure:[YOU FILL THIS IN.]

4. (Bollobas, problem 3.68, paraphrased.) Let G be a bipartite g raph with bipartition (A, B). Suppose that for $i \in \{1, 2\}$, M_i is a matching in G that matches $V_i \subseteq A$ to $W_i \subseteq B$. Prove that there is a matching M in G such that the subsets $V \subset A$ and $W \subseteq B$ of vertices matched by M satisfy $V_1 \subseteq V \subseteq V_1 \cup V_2$ and $W_2 \subseteq W \subseteq W_1 \cup W_2$.

5. (Bollobas, Problem 3.78.) Let G be a graph of minimal degree 3, without two edge-disjoint cycles. Show that G is one of the two graphs K_4 or $K_{3,3}$.

6. A graph is *critically k-edge-connected* if it is k-edge connected and after removing any edge it is no longer k-edge connected. Prove that any critically k edge-connected graph has a vertex of degree k. (Hint given below)

7. In this problem you will prove the correctness of a surprising algorithm for testing whether a bipartite graph has a one factor.

   For purposes of this problem, if $\bar{v}$ is a nonnegative nonzero real vector, *normalizing* $\bar{v}$ means to divide each entry of $\bar{v}$ by $\sum_i v_i$. Also, we say that a vector is *stochastic* if it is nonnegative and its sum is 1, and ϵ-stochastic if it is nonnegative and its sum is strictly between $1+\epsilon$ and $1/(1+\epsilon)$.

Let G be a bipartite graph with bipartition (V, W) with $|V| = |W| = n$. A $V \times W$ matrix M is *compatible with* G if $M[v, w] \neq 0$ implies $vw \in E$.

Let M be the $V \times W$ matrix with $M[v, w] = 1$ if $vw \in E$ and 0 otherwise. Consider the following algorithm. First let M_0 be obtained from M by normalizing each column. Then, for i from 1 to $10n^3 \log_2 n$ do: Let N_i be the matrix obtained by normalizing each row of M_{i-1}. Let M_i be the matrix obtained by normalizing each column of N_i.

In this problem you will prove:

Theorem. Let $M_f(G)$ be the final matrix produced by the algorithm. Then G has a one-factor and only if M_f is $1/n$-stochastic.

(a) Prove that if there is a matrix compatible with G whose columns are stochastic and whose rows are $1/n$-stochastic, then G has a one-factor.

(b) Recall that the permanent of a square matrix A, denoted $per(A)$ is the sum over all 1-1 mappings σ from the rows to the columns of $\prod_{i=1}^{n} A[i, \sigma(i)]$. Suppose that G has a one-factor. For each $i \geq 0$, prove that $per(M_{i+1}) \geq per(M_i)$. Furthermore prove that if $\epsilon \in [0, 1]$ and M_i is not ϵ-stochastic then $per(M_{i+1}) \geq (1+\frac{\epsilon^2}{8})per(M_i)$.

(c) Prove the Theorem.

Some hints

Problem 6 Consider a minimal subset A of vertices with the property that there are exactly k edges leaving A and prove that A must be a singleton. The statement proved in Assignment 2, problem 3 is useful here.