MATH 642:581–Spring, 2014

Assignment 2–Due February 19 (April 6 revision)

A few preliminary remarks.

1. Follow the general instructions for homework given in:

   http://www.math.rutgers.edu/∼saks/homework-grad.html

2. Some of these problems may be easier after the 1/29-1/30 lectures.
3. Please be on the look out for errors. If something seems not to make sense, check with me before investing a lot of time on the problem. I would appreciate being notified of any typos (even minor ones).

1. (Bollobas, Problem 3.11) For each choice of integers k, ℓ with $1 \leq k \leq \ell$ construct graphs $G_1 = G_1(k, \ell)$,$G_2 = G_2(k, \ell)$,$G_3 = G_3(k, \ell)$ such that:

   i. $\kappa(G_1) = k$ and $\lambda(G_1) = \ell$.

   ii. $\kappa(G_2) = k$ and $\kappa(G_2 - x) = \ell$ for some vertex x.

   iii. For some edge xy of G_3, $\lambda(G_3 - x) = k$ and $\lambda(G_3 - xy) = \ell$.

2. Let G be a k-connected graph with $k \geq 2$

   (a) Prove that every subset of k vertices belongs to a cycle.

   (b) Prove that the circumference of the graph is at least $\min(2k, |V(G)|)$.

3. The edge boundary of a set W of vertices of a graph G is the set of edges $E(W, V - W)$ that leave W. Prove that if G is a k-edge connected graph, then for any two subsets of vertices A and B such that $A \cap B \neq \emptyset$, and $A \cup B \neq V(G)$, if A and B both have edge boundary of size k then so do $A \cap B$ and $A \cup B$.

4. Bollobas, Problem 3.17. Let G be a connected graph with minimum degree $\delta(G) = k \geq 1$. Prove that G contains a path $x_1 \ldots x_k$ such that $G - \{x_1, \ldots, x_k\}$ is also connected. (A hint is given in the book, but is slightly misleading. The strategy proposed is on the right track, but you'll need to do a variant of what's suggested.)

5. (Diestel, 3.11)

   (a) Show that every cubic 3-edge connected graph is 3-connected.

   (b) Show that a graph is cubic and 3-connected if and only if it can be constructed from K_4 by successive applications of the following operation: subdivide two edges by inserting a new vertex on each of them and join the two new subdividing vertices by an edge. (Hint given below.)

6. For $k \geq 2$, let $\beta_k(n)$ be the maximum average degree among all graphs on n vertices which have no k-connected subgraph. Let $\gamma(k) = \limsup_{n \longrightarrow \infty} \beta_k(n)$. In class we showed (or will show) that $\gamma(k) \leq 4k$. Prove that $\gamma(k) \geq 3k - 4$. (Hint available below.)

7. Prove that a graph G with at least 3 vertices is 2-connected if and only if for any two vertices v, w there is an ordering of the vertices with v first and w last such that every vertex $u \notin \{v, w\}$ has a neighbor preceding u and a neighbor after u in the ordering. (Hint given below)

The final problems explore a beautiful connection between the connectivity of a graph and embeddings of the vertex set into $\mathbb{R}^n$ that have some special geometric properties.

We start by recalling some basic definitions from linear algebra and elementary analysis.

- If $x^1, \ldots, x^r$ are vectors in $\mathbb{R}^d$ an *affine combination* of $x^1, \ldots, x^r$ is a sum of the form $c_1 x^1 + \cdots + c_r x^r$ where $c_1, \ldots, c_r$ are real numbers summing to 1. A *convex combination* is an affine combination where all of the coefficients are nonnegative.

- For a subset $Z \subseteq \mathbb{R}^d$, the convex hull of Z, $conv(Z)$ is the set of all points that can be expressed as convex combinations of points in Z, and the affine hull of Z, $aff(Z)$ is the set of all points that can be expressed as an affine combination of points in Z.

- For a finite set X, $\mathbb{R}^X$ is the vector space of functions from X to $\mathbb{R}$. For $x \in X$, we write e^x for the function mapping x to 1 and all other elements to 0. Also Δ^X is the convex hull of $\{e^x : x \in X\}$ which is the set of functions $f \in \mathbb{R}^X$ with nonnegative coordinates summing to 1.

- If X is a finite set of points in $\mathbb{R}^d$, a point $x \in X$ is *extreme in* X if x is not in the convex hull of $X - \{x\}$. Let $E(X)$ be the set of points in X that are extreme in X. The following standard fact may be useful in what follows: X is a subset of $conv(E(X)$. (Exercise (not to hand in): prove this.)

Now let G be a connected graph.

- For $X \subseteq V(G)$, an X-embedding of G is a map f from V to Δ^X where for each $x \in X$ $f(x) = e^x$.

- An X-embedding is *neighborly* if for $v \in V - X$, $f(v)$ is in the convex hull of the set $\{f(w) : w \in N(v)\}$.

- An X-embedding is *generic* if for any set $\{w_1, \ldots, w_r\}$ of at most $|X|$ vertices, $f(w_1), \ldots, f(w_r)$ are linearly independent.

In the problems that follow you'll prove part of the following theorem: For $k \geq 2$, and any graph G with at least $k+1$ vertices, the following two conditions are equivalent: (1) G is k-connected and (2) for every subset X of k vertices there is a generic neighborly X-embedding of G. (Exercise (not to hand in): The case $k = 2$ of this theorem is equivalent to the result of problem 7.)

8. The goal of this problem is to prove (2) implies (1).

   (a) Let $X \subseteq V$ and suppose f is a 1-1 neighborly X-embedding of G. Let $W \subseteq V - X$ and let S be the set of vertices of $V - W$ that have a neighbor in W. Prove that for all $w \in W$, $f(w) \in conv(f(s) : s \in S)$.

   (b) Prove that (2) implies (1).

9. In this problem we'll make a step to proving (1) implies (2). Fix a subset X of $V(G)$. The aim is to give a general construction that gives a parameterized family of X-embeddings of G. In this problem we will not need to use the hypothesis that G is k-connected.

   Given an arbitrary function α that assigns a positive weight to every edge in G, we will construct an X-embedding g that depends on α. Let $S(\alpha)$ denote the following system of conditions on the function g: For every vertex $u \notin x$, $\sum_{v \in N(u)} \alpha(uv)(g(v) - g(u)) = 0$.

   (a) Prove that an X-embedding that satisfies $S(\alpha)$ is neighborly.

   (b) Next we prove that there is an X-embedding that satisfies $S(\alpha)$. For a function g, let $C_\alpha(g)$ be defined to be $\sum_{uv \in E(G)} \alpha(uv)\|g(u) - g(v)\|^2$. Show that among all X-embeddings g, $C_\alpha(g)$ attains a unique minimum and that this minimum satisfies system $S(\alpha)$.

10. (This is a difficult optional problem, which you may do in place of any two of the problems 1,2,3 or 4.) Finish the proof of (1) implies (2) by showing that if G is k-connected there is a way to choose α so that an X-embedding satisfying $S(\alpha)$ is generic.

Some hints

Problem 5 The following lemma may be helpful: Let G be a cubic 3-edge-connected graph and suppose there is at least one partition A, B of V such that $|A| > 1$ and $|B| > 1$ $|E(A, B)| = 3$. Let A be a subset of vertices of minimum size with the property that A is not a singleton set and $E(A, V - A)$ has size 3. Then the induced graph $G[A]$ has at least one edge, and for any edge xy in $G[A]$ the only edge cutsets of size 3 it belongs are the cutsets corresponding to the stars at x and y.

Problem 6 Give an explicit description (with proof) of a sequence $G_1, G_2, \ldots$ of graphs where $|V(G_1)| < |V(G_2)| < \cdots$ such that none of the graphs has a k-connected subgraph and the limit of the average degree approaches $3k - 4$.

Problem 7 Use the ear decomposition.