MATH 642:581–Spring, 2014
Assignment 1–Due February 5 (Version: February 2)

A few preliminary remarks.

1. Follow the general instructions for homework given in:

   http://www.math.rutgers.edu/∼saks/homework-grad.html

2. Some of these problems may be easier after the 1/29-1/30 lectures.
3. Please be on the look out for errors. If something seems not to make sense, check with me before investing a lot of time on the problem. I would appreciate being notified of any typos (even minor ones).

1. A sequence $\vec{d} = (d_1, \ldots, d_n)$ is said to be *connected-graphic* if there is a connected graph with degree sequence $\vec{d}$. State and prove a theorem that characterizes all such sequences.
2. Let H_n be the hypercube graph (whose vertices are vectors in $\{0,1\}^n$ with two vectors adjacent if they differ in exactly one coordinate). Determine (with proof) the radius, diameter, girth and circumference of H_n.
3. For a tree T and $\alpha \in [0,1]$, a vertex v is an α-balance point for T if every component of $T - v$ has at most $\alpha|V(T)|$ vertices. An edge e is an α-balance edge if both components of $T - e$ have size at most $\alpha|V(T)|$.
   (a) Determine with proof the smallest value α for which every tree has an α-balance point
   (b) Determine with proof the smallest value α for which every tree has an α-balance edge.
4. Prove that the only triangle free graph with n vertices and $\lfloor n^2/4 \rfloor$ edges is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. (Bollobas, problem 1.4.)
5. Prove that a regular bipartite graph of degree at least 2 has no bridge. (A bridge is an edge whose removal disconnects the graph.) (Bollobas, problem 1.24.)
6. (Bollobas, problem 1.16.)
   (a) Prove: Every nonempty graph has a vertex such that the average degree of its neighbors is at least the average degree of the entire graph.
   (b) Show that the above statement becomes false if we replace "at least" by "at most".
7. Prove that the following two properties of a graph are equivalent: G has an induced cycle of size at least 4, and G has an induced subgraph that has a minimal vertex cutset that is not a clique. (Remark: When we say that a set S is a *minimal* set satisfying condition C we mean that S satisfies condition C and no subset of S satisfies C. This is not the same as saying that S is a *minimum size set satisfying* C.)

8. Prove that if G is a graph that has no induced cycle of size at least 4, then it has a vertex v whose neighborhood is a clique. (Hint given below.)

Hints for Selected problems

Problem 8. Use induction and the result of the previous problem to prove the stronger statement: If G is not a clique and G has no induced cycle of size at least 4, then there are two non-adjacent vertices each of whose neighborhood is a clique.