Partition identities

Siddhartha Sahi
Rutgers University

December 9, 2007

1 Partitions

A partition λ of a positive integer n is a sequence of positive integers $n_1 \geq n_2 \geq \ldots \geq n_k \geq 1$ such that $n_1 + n_2 + \ldots + n_k = n$. The integers n_i are called the parts, and k is called the length of the partition. The number of partitions of n is written $p(n)$. For convenience we define $p(0) = 1$, i.e., we think of 0 as having one partition, the empty partition (of length 0).

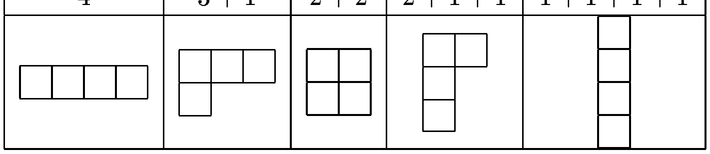
Here are the partitions for small n :

n	0	1	2	3	4	5
$p(n)$	1	1	2	3	5	7
	$\emptyset$	1	2 $1+1$	$3, 2+1$ $1+1+1$	$4, 3+1, 2+2$ $2+1+1$ $1+1+1+1$	$5, 4+1, 3+2, 3+1+1$ $2+2+1, 2+1+1+1$ $1+1+1+1+1$

The number $p(n)$ grows rapidly with n, e.g., $p(200) = 3,972,999,029,388$.

We represent a partition by a diagram consisting of rows of boxes as follows

4	$3+1$	$2+2$	$2+1+1$	$1+1+1+1$



If in the diagram of λ we count the boxes in successive *columns*, we get another partition λ' called the conjugate or transpose of λ. The diagram of λ' is obtained by transposing the diagram of λ (like a matrix), thus $(\lambda')' = \lambda$. For $n = 4$, the partition $2+2$ is self-conjugate, while the others pair up as follows $4 \leftrightarrow 1+1+1+1$, $3+1 \leftrightarrow 2+1+1$.

Let $p_m(n)$ be the number of partitions n with length $\leq m$, and let $\pi_m(n)$ be the number of partitions of n with largest part (hence all parts) $\leq m$.

Proposition 1 *For all integers m, n we have $p_m(n) = \pi_m(n)$.*

Proof. The function $p_m(n)$ counts those partitions of n whose diagrams have first column $\leq m$, while $\pi_m(n)$ counts those with first row $\leq m$. The two kinds of partitions are interchanged by the transpose map. ■

2 Generating functions

The study of partitions gives rise to a rich supply of sequences of the form $a_0, a_1, a_2, \ldots$; where a_n represents the number of partitions of n with some specified properties. To study a sequence, it is often convenient to look at the power series $a_0 + a_1 q + a_2 q^2 + \ldots$ where q is a variable. Sometimes one can use the ratio test to show that the series converges for some values of q, the resulting function $f(q)$ is called the generating function for the sequence. In some cases $f(q)$ this is related to a geometric series and we get a rational function of q. Here is a table of some simple q-series

sequence	$(1,1,1,1,\ldots)$	$(1,1,0,0,\ldots)$	$(1,0,1,0,1,0,\ldots)$
q-series	$1+q+q^2+\ldots$	$1+q$	$1+q^2+q^4+\ldots$
formula	$\frac{1}{1-q}, \lvert q\rvert < 1$	$1+q$, all q	$\frac{1}{1-q^2}, \lvert q\rvert < 1$

Many of the formulas that arise in the subject can be expressed more compactly in terms of the following notation

$$(x;q)_n := \prod_{i=1}^{n} \left(1 - q^{i-1}x\right) = (1-x)(1-qx)\ldots\left(1-q^{n-1}x\right)$$

If q is understood we will simply write $(x)_n$ for $(x;q)_n$.

We will also use $(x)_\infty$ for the infinite product $\prod_{i=1}^{\infty}\left(1-q^{i-1}x\right)$. Some care is required in dealing with infinite products, see Appendix B in Andrews for details on convergence issues. However we can understand this "formally", i.e., without worrying about convergence issues. Simply note that for each n the coefficient of q^n is the same as in the finite product $(x)_{n+1}$

If S is some set of integers, we write $p(S,n)$ for the number of partitions of n all of whose parts belong to S.

Proposition 2 *The generating function of* $p(S,n)$ *is* $\prod_{s\in S}\frac{1}{1-q^s}$

Proof. We expand each factor $\frac{1}{1-q^s}$ as a geometric series $1+q^s+q^{2s}+\ldots$. Now if the elements of S are $s_1, s_2, \ldots$ then the product can be written as follows:

$$\left(1+q^{s_1}+q^{2s_1}+\ldots\right)\left(1+q^{s_2}+q^{2s_2}+\ldots\right)\ldots$$

If we multiply this out then we get terms of the form $q^{k_1 s_1 + k_2 s_2 + \cdots}$. These terms correspond precisely to all possible partitions with parts from S. So each q^n will show up $p(S,n)$ times in the expanded product. ■

Corollary 3 *The generating function of* $\pi_m(n) = p_m(n)$ *is* $\frac{1}{(q)_m}$; *and that of* $p(n)$ *is* $\frac{1}{(q)_\infty}$.

Now for each k we define $D_k(n)$ to be the number of partitions where successive parts differ by at least k. Thus $D_0(n) = p(n)$, while $D_1(n)$ is the number of partitions of n into *distinct* parts. The generating function of $D_1(n)$ is $\prod_{i=1}^{\infty}\left(1+q^i\right)$, as is easily seen by multiplying out the infinite product. The following result is called Euler's partition theorem.

Theorem 4 *The number of partitions of n with distinct parts is the same as those with odd parts. Thus $D_1(n) = p(S, n)$ where $S = \{1, 3, 5, \ldots\}$.*

Proof. The generating function for $p(S, n)$ is

$$\prod_{i \text{ odd}} \frac{1}{1-q^i} = \frac{\prod_{i=1}^{\infty}\left(1-q^{2i}\right)}{\prod_{i=1}^{\infty}\left(1-q^{i}\right)} = \frac{\prod_{i=1}^{\infty}\left(1-q^{i}\right)\left(1+q^{i}\right)}{\prod_{i=1}^{\infty}\left(1-q^{i}\right)} = \prod_{i=1}^{\infty}\left(1+q^{i}\right)$$

which is the same as for $D_1(n)$. ■

We now determine a generating function for $D_k(n)$. More generally we define $D_k^l(n)$ to be the number of partitions of n with successive differences $\geq k$, and smallest part $\geq l$; and we further define $D_k^{l,m}(n)$ to be the number of such partitions with length exactly m. Thus we have

$$D_k^l(n) = \sum_m D_k^{l,m}(n) \text{ and } D_k(n) = D_k^1(n)$$

Proposition 5 *The generating function of $D_k^{l,m}(n)$ is $q^{lm+mk(k-1)/2}/(q)_m$.*

Proof. Suppose λ is a partition with parts $n_1 \geq n_2 \geq \ldots \geq n_m$ satisfying the conditions of $D_k^{l,m}(n)$, and define

$$d_i = n_i - (m-i)k - l$$

Then $d_1 \geq d_2 \geq \ldots \geq d_m \geq 0$ is a partition of $n-a$ for $a = lm + mk(k-1)/2$. Conversely given a partition of $n-a$ of length $\leq m$, we can add $(m-i)k + l$ to the parts to obtain a partition satisfying the conditions of $D_k^{l,m}(n)$. Thus we have $D_k^{l,m}(n) = 0$ for $n < a$, and $D_k^{l,m}(n) = \pi_m(n-a)$ for $n \geq a$. Hence the generating series of $D_k^{l,m}$ is

$$\sum_{n\geq a} \pi_m(n-a)q^n = \sum_{n\geq 0} \pi_m(n)q^{n+a} = q^a/(q)_m$$

by the previous corollary. ■

Corollary 6 *The generating function for $D_k^l(n)$ is $\sum_{m=0}^{\infty} q^{lm+mk(k-1)/2}/(q)_m$.*

3 The product formulas

The q-binomial theorem is the following formula for the product $(x)_n$.

Theorem 7 *We have*

$$(x)_n = \sum_{k=0}^{n} (-1)^k \frac{(q)_n q^{k(k-1)/2}}{(q)_{n-k}(q)_k} x^k$$

Proof. Let $(x)_n = \sum_{k=0}^{n} a_k x^k$. Since $(1-q^n x)(x)_n = (1-x)(qx)_n$ we get

$$(1-q^n x)\sum_{k=0}^{n} a_k x^k = (1-x)\sum_{k=0}^{n} a_k q^k x^k$$

and equating the coefficients of x^k on the two sides we get the relation

$$a_k - q^n a_{k-1} = q^k a_k - q^{k-1} a_{k-1}.$$

We can rewrite this as $a_k = b_k a_{k-1}$ where

$$b_k = \frac{q^n - q^{k-1}}{1-q^k} = -q^{k-1}\frac{1-q^{n-k+1}}{1-q^k}.$$

Iterating this k times we get $a_k = b_k b_{k-1} a_{k-2} = \ldots = b_k b_{k-1} \ldots b_1 a_0$, i.e.,

$$a_k = (-1)^k q^{(k-1)+\cdots+2+1}\frac{\left(1-q^{n-k+1}\right)\ldots(1-q^n)}{(1-q^k)\ldots(1-q)} a_0 = (-1)^k \frac{(q)_n q^{k(k-1)/2}}{(q)_{n-k}(q)_k} a_0$$

Since $a_0 = 1$, the result follows. ■

Corollary 8 *We have*

$$(x)_\infty = \sum_{k=0}^{n} (-1)^k \frac{q^{k(k-1)/2}}{(q)_k} x^k.$$

Proof. As $n \to \infty$, $\frac{(q)_n}{(q)_{n-k}} \to \frac{(q)_\infty}{(q)_\infty} = 1$. Hence $a_k \to (-1)^k \frac{q^{k(k-1)/2}}{(q)_k}$. ■

There is also a related double product formula.

Proposition 9 *We have*

$$(x)_n (q/x)_n = \sum_{j=-n}^{n} (-1)^j \frac{(q)_{2n} q^{j(j-1)/2}}{(q)_{n+j}(q)_{n-j}} x^j.$$

Proof. We first rewrite $(q/x)_n$ as follows

$$(q/x)_n = (1-q/x)\ldots(1-q^n/x) = c_n\left(1-q^{-1}x\right)\ldots\left(1-q^{-n}x\right)$$

where $c_n = (-1)^n q^{n(n+1)/2} x^{-n}$. Now using the q-binomial theorem we get

$$\begin{aligned}
(x)_n (q/x)_n = c_n \left(q^{-n}x\right)_{2n} &= c_n \sum_{k=0}^{2n} (-1)^k \frac{(q)_{2n} q^{k(k-1)/2}}{(q)_{2n-k}(q)_k} x^k \\
&= c_n \sum_{j=-n}^{n} (-1)^{j+n} \frac{(q)_{2n} q^{(j+n)(j+n-1)/2}}{(q)_{n+j}(q)_{n-j}} \left(q^{-n}x\right)^{j+n} \\
&= \sum_{j=-n}^{n} (-1)^j \frac{(q)_{2n} q^{a(n,j)}}{(q)_{n+j}(q)_{n-j}} x^j
\end{aligned}$$

where $a(n,j) = \frac{1}{2}[n(n+1)+(j+n)(j+n-1)-2n(j+n)] = \frac{1}{2}j(j-1)$. ■

Taking the limit $n \to \infty$ yields the Jacobi triple product identity.

Theorem 10 *We have*

$$(q)_\infty (x)_\infty (q/x)_\infty = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(j-1)/2} x^j.$$

Proof. We let $n \to \infty$ in the previous proposition; then $(q)_{2n}, (q)_{n+j}(q)_{n-j}$ all tend to $(q)_\infty$ and the result follows. ■

4 The Rogers-Ramanujan identities

The Rogers-Ramanujan identities are the following two formulas

Theorem 11 *We have*

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_j} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \text{ and } \sum_{j=0}^{\infty} \frac{q^{j^2+j}}{(q)_j} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

By Corollary 6 and Proposition 2, these can be intrepreted as statements about partitions. The left sides are generating functions of $D_2^1(n)$ and $D_2^2(n)$, while the right sides are $\prod \frac{1}{1-q^s}$ for $\{s \equiv 1,4 \bmod 5\}$ and $\{s \equiv 2,3 \bmod 5\}$ respectively. Thus we deduce that for each n, the number of partitions with successive differences ≥ 2 is the same as those with parts congruent to $1,4$ mod 5; and the number of partitions with successive differences ≥ 2 and smallest part ≥ 2, is the same as those with parts congruent to $2,3 \bmod 5$.

For the proof we first reexpress the right sides of the two identities as sums.

Lemma 12 *The following identities hold*

$$\frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} = \sum_{j=-\infty}^{\infty} \frac{(-1)^j q^{(5j^2-j)/2}}{(q)_\infty}$$
$$\frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = \sum_{j=-\infty}^{\infty} \frac{(-1)^j q^{(5j^2-3j)/2}}{(q)_\infty}.$$

Proof. For the first identity we note that

$$\frac{(q)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} = (q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5).$$

We now use the Jacobi triple product identity with $q = q^5$ and $x = q^2$ to express the right side as

$$\sum_{j=-\infty}^{\infty} (-1)^j (q^5)^{j(j-1)/2} (q^2)^j = \sum_{j=-\infty}^{\infty} (-1)^j q^{(5j^2-j)/2}.$$

This proves the first identity, and the second is similar. ■

The main ingredient in the proof of Theorem 11 is the following.

Definition 13 *Two sequences* $\{a_n\}_{n=-\infty}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *form a Bailey pair if*

$$b_n = \sum_{j=-n}^{n} \frac{a_j}{(q)_{n+j}(q)_{n-j}}.$$

The key fact about such pairs is the following result known as the Bailey lemma, which will be proved in the next section.

Theorem 14 *If* $\{a_n\}, \{b_n\}$ *are a Bailey pair then so are* $\left\{a_n q^{n^2}\right\}, \left\{\sum_{j=0}^{n} \frac{b_j q^{j^2}}{(q)_{n-j}}\right\}$.

We can now complete the proof of Rogers-Ramanujan identities.

Proof. (of Theorem 11) By Proposition 9 we have

$$\frac{(x)_n (q/x)_n}{(q)_{2n}} = \sum_{j=-n}^{n} \frac{(-1)^j q^{(j^2-j)/2} x^j}{(q)_{n+j}(q)_{n-j}}.$$

This means that the following sequences form a Bailey pair for each x.

$$\left\{(-1)^n q^{(n^2-n)/2} x^n\right\}_{n=-\infty}^{\infty}, \left\{\frac{(x)_n (q/x)_n}{(q)_{2n}}\right\}_{n=0}^{\infty} \tag{1}$$

If we specialize $x = 1$ the right sequence becomes $\{\delta_{n,0}\} = 1, 0, 0, \ldots$. Repeatedly applying the Bailey lemma we get the following chain of Bailey pairs

a_n	$(-1)^n q^{(n^2-n)/2}$	$(-1)^n q^{(3n^2-n)/2}$	$(-1)^n q^{(5n^2-n)/2}$
b_n	$\delta_{n,0}$	$1/(q)_n$	$\sum_{j=0}^{n} \frac{q^{j^2}}{(q)_j (q)_{n-j}}$

The last column now gives the Bailey relation

$$\sum_{j=0}^{n} \frac{q^{j^2}}{(q)_j (q)_{n-j}} = \sum_{k=-n}^{n} (-1)^k \frac{q^{(5k^2-k)/2}}{(q)_{n+k}(q)_{n-k}}$$

and as $n \to \infty$, we obtain

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_j (q)_\infty} = \sum_{k=-\infty}^{\infty} (-1)^k \frac{q^{(5k^2-k)/2}}{(q)_\infty (q)_\infty}.$$

and applying Lemma 12 we obtain the first Rogers-Ramanujan identity.

For $x = q^{-1}$, the right sequence in (1) is $\left\{\delta_{n,0} - q^{-1}\delta_{n,1}\right\} = 1, -q^{-1}, 0, \ldots$ Since

$$\sum_{j=0}^{n} \left(\delta_{j,0} - q^{-1}\delta_{j,1}\right) \frac{q^{j^2}}{(q)_{n-j}} = \frac{1}{(q)_n} - \frac{1}{(q)_{n-1}} = \frac{q^n}{(q)_n}$$

we get the following chain of Bailey pairs

a_n	$(-1)^n q^{(n^2-3n)/2}$	$(-1)^n q^{(3n^2-3n)/2}$	$(-1)^n q^{(5n^2-3n)/2}$
b_n	$\delta_{n,0} - q^{-1}\delta_{n,1}$	$q^n/(q)_n$	$\sum_{j=0}^{n} \frac{q^{j^2+j}}{(q)_j (q)_{n-j}}$

.

The second Rogers-Ramanujan identity follows from the last column. ■

5 The WZ method and the Bailey Lemma

Let $f(n,j)$ be a function of two integer variables n,j, and consider the sum

$$\sum_{j=-\infty}^{\infty} f(n,j).$$

For simplicity we assume $f(n,j)$ has *finite support in* j i.e., for each n we have $f(n,j)=0$ for all except finitely many values of j.

Now suppose we have a conjectured formula $S(n)$ for the sum, which we want to prove. Writing $F(n,j)$ for $f(n,j)/S(n)$, we need to show

$$\sum_{j=-\infty}^{\infty} F(n,j) = 1.$$

If we can prove that this holds for some $n=n_0$, then it remains only to prove that $\sum_j F(n,j)$ is independent of n. The following lemma due to H. Wilf and D. Zeilberger, provides a method for accomplishing this.

Lemma 15 *If there is a function $G(n,j)$ with finite support in j such that*

$$F(n+1,j) - F(n,j) = G(n,j+1) - G(n,j) \text{ for all } n,j \tag{2}$$

then $\sum_{j=-\infty}^{\infty} F(n,j)$ is independent of n.

Proof. Summing (2) over j we obtain

$$\sum_{j=-\infty}^{\infty} F(n+1,j) - \sum_{j=-\infty}^{\infty} F(n,j) = \sum_{j=-\infty}^{\infty} G(n,j+1) - \sum_{j=-\infty}^{\infty} G(n,j) = 0$$

Thus we get $\sum_j F(n+1,j) = \sum_j F(n,j)$, and the independence follows. ∎

We now apply these ideas to prove the Bailey lemma (Theorem 14). We first prove the following result, which is really a special case.

Proposition 16 *For all $n \ge k \ge 0$, we have*

$$\sum_{j=k}^{n} \frac{q^{j^2}}{(q)_{n-j}(q)_{j+k}(q)_{j-k}} = \frac{q^{k^2}}{(q)_{n+k}(q)_{n-k}}.$$

Proof. Dividing by the RHS it suffices to prove $\sum_{j=-\infty}^{\infty} F(n,j) = 1$, where

$$F(n,j) = \begin{cases} \dfrac{q^{j^2-k^2}}{(q)_{n-j}} \dfrac{(q)_{n+k}(q)_{n-k}}{(q)_{j+k}(q)_{j-k}} & \text{if } k \le j \le n \\ 0 & \text{otherwise} \end{cases}.$$

For $n=k$, the sum reduces to the single term 1 so the result holds in this case. To finish the proof we will show $G(n,j) = -q^{n+j}F(n,j-1)$ satisfies (2).

For this we must prove that for all n, j

$$F(n+1,j) - F(n,j) = -q^{n+j+1} F(n,j) + q^{n+j} F(n,j-1)$$

or equivalently that

$$F(n+1,j) - q^{n+j} F(n,j-1) = \left(1 - q^{n+j+1}\right) F(n,j) \tag{3}$$

If $j > n$ or $j < k$ then both sides of (3) are 0. If $k \le j \le n$, then we have

$$\begin{aligned} F(n+1,j) &= \frac{\left(1-q^{n+k+1}\right)\left(1-q^{n-k+1}\right)}{\left(1-q^{n-j+1}\right)} F(n,j) \\ F(n,j-1) &= q^{-2j+1} \frac{\left(1-q^{j+k}\right)\left(1-q^{j-k}\right)}{\left(1-q^{n-j+1}\right)} F(n,j) \end{aligned}$$

Now (3) reduces to the following simple identity

$$\begin{aligned} &\left(1-q^{n+k+1}\right)\left(1-q^{n-k+1}\right) - q^{n-j+1}\left(1-q^{j+k}\right)\left(1-q^{j-k}\right) \\ &= \left(1-q^{n-j+1}\right)\left(1-q^{n+j+1}\right) \end{aligned}$$

whose verification we leave to the reader. ■

We can now finish the proof of the Bailey lemma.

Proof. (of Theorem 14) We need to prove that

$$b_j = \sum_{k=-j}^{j} \frac{a_k}{(q)_{j+k}(q)_{j-k}} \implies \sum_{j=0}^{n} \frac{b_j q^{j^2}}{(q)_{n-j}} = \sum_{k=-n}^{n} \frac{a_k q^{k^2}}{(q)_{n+k}(q)_{n-k}}$$

For this we calculate as follows

$$\begin{aligned} \sum_{j=0}^{n} \frac{b_j q^{j^2}}{(q)_{n-j}} &= \sum_{j=0}^{n} \sum_{k=-j}^{j} \frac{a_k}{(q)_{j+k}(q)_{j-k}} \frac{q^{j^2}}{(q)_{n-j}} \\ &= \sum_{k=-n}^{n} a_k \left[\sum_{j=|k|}^{n} \frac{q^{j^2}}{(q)_{n-j}(q)_{j+k}(q)_{j-k}} \right] = \sum_{k=-n}^{n} \frac{a_k q^{k^2}}{(q)_{n+k}(q)_{n-k}} \end{aligned}$$

where the last equality follows by Proposition 16. ■