

# Math 581, Homework 3, due 2/28/06

Radoš Radoičić

February 17, 2006

**You are required to hand in any 9 of the following problems.**

**Problems 1 – 8 from the book (electronic edition):**

Chapter 4 – Problems: 1, 2, 4, 5, 16, 18, 19, 20.

**Problem 9:** Let  $G$  be a plane graph, whose exterior face is bounded by a cycle  $u_1, u_2, \dots, u_k$  ( $k \geq 4$ ). An edge of  $G$  connecting two non-consecutive vertices along this cycle is called a chord. Prove that there are at least two non-consecutive vertices that are not incident to any chord.

**Problem 10: Rosenstiehl-Tarjan theorem** Prove the following statement: The vertices and the edges of any planar graph can be represented by horizontal and vertical segments, respectively, such that 1) no two segments have an interior point in common; and 2) two horizontal segments are connected by a vertical segment if and only if the corresponding vertices are adjacent. (Hint: Use the canonical labeling for triangulations.)

**Problem 11:** There are  $n$  convex sets in the plane such that the boundaries of any two of them have at most two common points. Show that the boundary of their union consists of at most  $6n - 12$  connected arcs of the boundaries of the sets.

**Problem 12:** Let  $\{p_1, p_2, \dots, p_n\}$  be a set of  $n \geq 3$  points in the plane such that the minimum distance between them is at least one. Show that the number of pairs  $\{p_i, p_j\}$  at distance exactly one is at most  $3n - 6$ . Can this bound be improved?

**Problem 13: Difficult** Let  $l_1, l_2, \dots, l_n$  be distinct horizontal lines in the plane. Show that every planar graph on  $n$  vertices has a straight-line embedding such that 1) every vertex is mapped into a point of  $l_1 \cup l_2 \cup \dots \cup l_n$ ; and 2) no two vertices are mapped into the same line  $l_i$ .

**Problem 14:** Draw the complete bipartite graph  $K^{n,n}$  in the plane with at most  $n^4/16$  crossings. Avoid cheating, i.e. every crossing involves only two edges. Also, draw  $K^7$  with 9 crossings. Show that you need to have at least 7 crossings.

**Problem 15:** You are given an arrangement of  $n$  lines in the plane, such that no two lines are parallel and no three lines go through the same point. Introduce a vertex at each crossing and consider the plane graph of the bounded edges. Calculate the number of edges in this graph. Show that it is 2-connected. Prove that its chromatic number is 3.

**Problem 16:** A *claw* is a bipartite graph  $K^{1,3}$ . A claw covering of graph  $G$  is a partition of all edges of  $G$  into claws. Prove that every plane triangulation  $\Gamma$  contains a face  $\Delta$ , so that  $\Gamma - \Delta$  has a claw covering.

**Problem 17:** Let  $G$  be a triangulation with at least four vertices. Letting  $n_i$  be the number of vertices of degree  $i$ , prove that  $3n_3 + 2n_4 + n_5 \geq 12$ .

**Problem 18: Pick's formula** For a near-triangulation  $G$ , let  $B$  be the number of vertices on the exterior face, and let  $I$  be the number of vertices in the interior, so that  $G$  has  $I + B$  vertices in total. a) Determine the numbers of edges and faces in  $G$ . b) Let  $T$  be a triangle with corners at integer lattice points in the plane. Prove that if no other lattice points lie on the boundary or in the interior of  $T$ , then  $T$  has area  $1/2$ . (Hint: e. g. use induction on the perimeter of the smallest lattice rectangle containing  $T$ .) c) A lattice polygon is a polygon whose corners are at integer lattice points in the plane. Let  $I$  be the number of lattice points in the interior and  $B$  be the number of lattice points on the boundary for a lattice polygon  $P$ . Prove Pick's formula, which states that the area of the region bounded by  $P$  is  $I + B/2 - 1$ .

**Problem 19:** Let  $G$  be the graph obtained from the 3-dimensional cube  $Q_3$  by adding an edge with endpoints  $(0, 0, 0)$  and  $(1, 1, 1)$ . Find a planar embedding of  $G$  or show that it is nonplanar by using Kuratowski's Theorem.

**Problem 20: A planarity criterion (Hanani-Tutte)** a) Prove that in every drawing of  $K^5$  or  $K^{3,3}$  in the plane, there are two nonincident edges that cross an odd number of times. (Hint: Prove that the number of such pairs is always odd.) b) Prove that if a graph  $G$  has a drawing in which every two nonincident edges cross an even number of times, then  $G$  is planar. (Hint: You may use part a))