SOLUTIONS—ASSIGNMENT 7

Chapter 3. Problems: p. 111 #84. This is gambler's ruin with $n = 30$, $i = 15$ and $p = 0.55$, so the probability of winning is $\dfrac{1-(0.45/0.55)^{15}}{1-(0.45/0.55)^{30}} = \dfrac{1}{1+(9/11)^{15}} \approx 0.953026559$.

p. 111 #85. (a) Let S_k denote the event of two heads on turn k, $F_k = S_k^c$. After F_1F_2 we are back where we started, so

$$\begin{aligned} P(A) &= P(A|S_1)P(S_1) + P(A|F_1S_2)P(F_1S_2) + P(A|F_1F_2)P(F_1F_2) \\ &= P_1^2 + 0 + P(A)(1-P_1^2)(1-P_2^2); \end{aligned}$$

solving this for $P(A)$ gives the answer in the text. (c) is similar.

(b) Let Q_{nm} be the probability that A wins if he needs n heads, B needs m heads, and A has the coin; let R_{nm} denote the corresponding probabilities when B has the coin. Conditioning on the result of the first flip gives

$$Q_{nm} = P_1Q_{n-1,m} + (1-P_1)R_{nm}, \qquad R_{nm} = P_2R_{n,m-1} + (1-P_2)Q_{nm},$$

so

$$Q_{nm} = \frac{P_1Q_{n-1,m} + (1-P_1)P_2R_{n,m-1}}{P_1+P_2-P_1P_2}, \qquad R_{nm} = \frac{P_2R_{n,m-1} + (1-P_2)P_1Q_{n-1,m}}{P_1+P_2-P_1P_2}$$

Then starting from $Q_{n0} = R_{n0} = 0$, $Q_{0m} = R_{0m} = 1$ we may compute everything: setting $D = P_1 + P_2 - P_1P_2$ for convenience we have $Q_{11} = P_1/D$, $R_{11} = P_1(1-P_2)/D$, $Q_{12} = P_1[D+P_2(1-D)]/D^2$, $R_{21} = (1-P_2)P_1^2/D^2$, and $Q_{22} = P_1^2[D+2P_2(1-D)]/D^3$. Q_{22} is the answer to (b), continuing we could find Q_{33}, the answer to (d).

p. 111 #87. (a) Condition on the relevant events for the first 3 withdrawals. Since the probability that a white ball is obtained on a given withdrawal is $4/12 = 1/3$, we obtain for the probability p_1 that player A wins the equation $p_1 = (1/3) + (2/3)^3p_1$, and more generally for the probability p_i that player i wins the equation $p_i = (1/3)(2/3)^{i-1} + (2/3)^3p_i$, with solution $p_i = 9(2/3)^{i-1}/19$. So if A is the event that A wins, etc., we have $P(A) = 9/19$, $P(B) = 6/19$, $P(C) = 4/19$.

(b) Let W denote drawing a white ball, O denote drawing a ball of some other color. A wins if the draw is W or $OOOW$ or $OOOOOOW$, so

$$P(A) = \frac{4}{12} + \frac{8\cdot7\cdot6\cdot4}{12\cdot11\cdot10\cdot9} + \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot4}{12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6} = \frac{7}{15}.$$

$P(B)$ and $P(C)$ are calculated in the same way.

Chapter 4. Problems: p. 175 #1. Using unordered samples, we have these possiblitites:

Outcome	Value of X	*Probability*
WW	-2	$\binom{8}{2}/\binom{14}{2}$
WB	1	$(8)(4)/\binom{14}{2}$
WO	-1	$(8)(2)/\binom{14}{2}$
BB	4	$\binom{4}{2}/\binom{14}{2}$
BO	2	$(4)(2)/\binom{14}{2}$
OO	0	$\binom{2}{2}/\binom{14}{2}$

p. 175 #4.

$$\begin{aligned}
P(X=1) &= \frac{5\cdot 9!}{10!} = \frac{5}{10} = \frac{1}{2}\\
P(X=2) &= \frac{5\cdot 5\cdot 8!}{10!} = \frac{25}{90} = \frac{5}{18}\\
P(X=3) &= \frac{5\cdot 4\cdot 5\cdot 7!}{10!} = \frac{100}{10\cdot 9\cdot 8} = \frac{5}{36}\\
P(X=4) &= \frac{5\cdot 4\cdot 3\cdot 5\cdot 6!}{10!} = \frac{300}{10\cdot 9\cdot 8\cdot 7} = \frac{5}{84}\\
P(X=5) &= \frac{5\cdot 4\cdot 3\cdot 2\cdot 5\cdot 5!}{10!} = \frac{600}{10\cdot 9\cdot 8\cdot 7\cdot 6} = \frac{5}{252}\\
P(X=6) &= \frac{5!\cdot 5!}{10!} = \frac{1}{252}\,.
\end{aligned}$$

$P(X=k)=0$ for $k=7,\,8,\,9,\,10$. In more detail, there are 10! possible rankings of all examinees. If the top woman's rank is i then certainly $i \le 6$, *i.e.*, $P\{X=i\}=0$ for $i \ge 7$. To get $X=i$ we must choose men for the first $i-1$ places, in one of $(5)(4)\cdots(5-i+2)$ ways, then a woman in one of 5 ways, then the rest of the $10-i$ examinees; thus

$$P\{X=i\} = \frac{(5)(4)\cdots(5-i+2)(5)(10-i)!}{10!} = \frac{(5)(4)\cdots(5-i+2)(5)}{(10)(9)\cdots(10-i+1)}$$

(For $i=1$, the product $(5)(4)\cdots(5-i+2)$ should be understood as 1.)
Perhaps more simply, think of choosing the rankings as sampling without replacement. Then $X=1$ amounts to choosing a girl the first time, which has probability $5/10$. $X=2$ amounts to choosing a boy and then a girl, so that the probability for this is $\frac{5}{10}\cdot\frac{5}{9}$ and similarly for the rest. Note also that $P(X=6)=\frac{1}{\binom{10}{5}}$. Why?

p. 175 #5. Possible values of X are n (all heads), $n-2$ ($n-1$ heads, 1 tail), $n-4$ ($n-2$ heads, 2 tails), …, $-n+2$, $-n$ (all tails).

p. 175 #6. The sample space S is all strings of length 3 with letters H and T. $P\{X=3\} = P\{HHH\} = (1/2)^3 = P\{X=-3\}$, $P\{X=1\} = P\{\ 2H\&1T\} = \binom{3}{2}(1/2)^3 = P\{X=-1\}$.

p. 175 #7. (a) $\{1,\,2,\,3,\,4,\,5,\,6\}$ (b) same. (c) $\{2,\,3,\,4,\ldots,\,11,\,12\}$
(d) $\{-5,\,-4,\,-3,\,-2,\,-1,\,0,\,1,\,2,\,3,\,4,\,5\}$

p. 175 #8.
(a) $\frac{1}{36}, \frac{4-1}{36}, \frac{9-4}{36}, \frac{16-9}{36}, \frac{25-16}{36}, \frac{36-25}{36}$
(b) $\frac{11}{36}, \frac{9}{36}, \frac{7}{36}, \frac{5}{36}, \frac{3}{36}, \frac{1}{36}$
(c) $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}.$
(d) $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}.$

The various parts are pretty much the same; let's work out (b). Let X be the minimum appearing. $X=6$ if we roll 6–6; probability 1/36. $X=5$ if we roll 5–5, 5–6, or 6–5; probability 3/36. $X=4$ if we roll 4–4, 4–5, 4–6, 5–4, or 6–4; probability 5/36. Similarly, there are 7 ways to get $X=3$, 9 to get $X=2$, and 11 to get $X=1$, for probabilities 7/36, 1/4, and 11/36, respectively.

p. 176 #14. Hint: Let Y_i be the number assigned to player i. Note that $\{X=0\}=\{Y_1<Y_2\}$, $\{X=1\}=\{Y_3>Y_1>Y_2\}$, $\{X=2\}=\{Y_4>Y_1>\max(Y_2,Y_3)\}$, …, and $\{X=4\}=\{Y_1>\max(Y_2,Y_3,Y_4,Y_5)\}$.