SOLUTIONS—ASSIGNMENT 6

Chapter 3. Problems: p. 107 #51a. The events are: C: has cancer; H healthy; Y: test says cancer.

$$P(C|Y) = \frac{P(Y|C)P(C)}{P(Y|C)P(C) + P(Y|H)P(H)} = \frac{(.268)(.7)}{(.268)(.7) + (.135)(.3)}.$$

p. 108 #60. Suppose we have N sophomore girls. Then if a student is selected at random, and F denotes freshman, B boy, then $P(FB) = 4/(16+N)$ and $P(F) = P(B) = 10/(16+N)$, so F and B are independent if $4/(16+N) = \big[10/(16+N)\big]^2$, or $N = 9$.

p. 108 #65. From the information given, both of Smith's parents have a brown-eyed and a blue-eyed gene. Moreover, because Smith has brown eyes, the reduced sample space, of equally likely outcomes, corresponding to this information is $\{(brown, blue), (blue, brown), (brown, brown)\}$, where the first entry specifies the gene from Smith's mother and the second from his father. It follows that the answer to (a) is 2/3. For (b), condition on whether Smith has 1 or 2 brown genes. Then writing F for the event that Smith's first child has blue eyes, we have $P(F) = P(F|\,1\,)P(1) + P(F|\,2\,)P(2) = \frac{1}{2}\cdot\frac{2}{3} + 0 = 1/3$. (c) Let E_i be the event that Smith's ith child has brown eyes. Then $P(E_2|E_1) = P(E_1E_2)/P(E_1) = \frac{1}{4}\cdot\frac{2}{3} + 1\cdot\frac{1}{3}/\frac{2}{3} = 3/4$, where $P(E_1E_2)$ has been computed by conditioning as in (b), using the fact that $P(E_1E_2\,|\,1\,) = (\frac{1}{2})^2$.

p. 110 #76: **(a)** The event that all are boys has probability $(\dfrac{1}{2})^5 = \dfrac{1}{32}$, the event that all are girls has the same probability, and since the two events are mutually exclusive the probability that all are of the same sex is 1/16. **(b)** The conditions specify 5 independent events of probability 1/2 each, so they specify an event of probability $(\dfrac{1}{2})^5 = \dfrac{1}{32}$. **(c)** Since there are $\binom{5}{3}$ ways to choose in which 3 of the 5 trials the boys occur, the probability is $\dbinom{5}{3}(\dfrac{1}{2})^5 = 10\cdot\dfrac{1}{32} = \dfrac{5}{16}$. **(d)** Only the first two tries are important; the probability is $(1/2)^2 = 1/4$. **(e)** This is the complement of the event that all are boys, with probability $1 - 1/32 = 31/32$.

p. 110 #77. Hint: Condition on the result of the first two rolls of the pair of dice and solve the equation thereby obtained for the desired probability $P(A)$. The equation is

$$P(A) = \frac{1}{9} + P(A)(1-\frac{1}{9})(1-\frac{5}{36}),$$

yielding $P(A) = 9/19$.

p. 110 #82. Since any roll not yielding a 7 or an even number is irrelevant we may think of the experiment as a sequence of independent trials as follows: one trial consists of rolling until either a 7 (success) or an even number (failure) appears. Now $P(7) = 6/36 = 1/6$ and $P(\text{Even}) = 18/36 = 1/2$, so that by a problem worked in class (see end of Example 4h) the probability of rolling a 7 before an even number—that is, the probability of success in one of our new trials—is 1/4. Now we are reduced to the problem of points: we want the probability of 2 successes before 6 failures. This is just the probability of at least 2 success in the first 7 trials:

$$\sum_{k=2}^{7}\binom{7}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{7-k} = \frac{21\cdot 3^5 + 35\cdot 3^4 + 35\cdot 3^3 + 21\cdot 3^2 + 7\cdot 3 + 1}{4^7}.$$

Theoretical exercises: p. 113 #6. If we accept the fact that the independence of $E_1,\ldots,E_n$ implies also the independence of $E_1^c,\ldots,E_n^c$ then the result is immediate, by DeMorgan's laws:

$$P\left(\bigcup E_k\right) = P\left(\left(\bigcap E_k^c\right)^c\right) = 1 - P\left(\bigcap E_k^c\right) = 1 - \prod P(E_k^c) = 1 - \prod(1-P(E_k)).$$

We may prove the independence of the complements (actually something a bit stronger) as follows. Suppose that $k_1, \ldots, k_r, j_1, \ldots, j_s$ are distinct indices between 1 and n; we prove that

$$P(E_{k_1} \cap \cdots \cap E_{k_r} \cap E_{j_1}^c \cap \cdots \cap E_{j_s}^c) = P(E_{k_1}) \cdots P(E_{k_r}) P(E_{j_1}^c) \cdots P(E_{j_s}^c). \tag{$*$}$$

The proof is by induction on s; for $s = 0$, $(*)$ is just the independence of the E_k. We assume that $(*)$ holds for s and check it for $s+1$. Now

$$\begin{aligned} E_{k_1} \cap \cdots \cap E_{k_r} \cap E_{j_1}^c \cap \cdots \cap E_{j_s}^c = {}& [E_{k_1} \cap \cdots \cap E_{k_r} \cap E_{j_1}^c \cap \cdots \cap E_{j_s}^c \cap E_{j_{s+1}}] \\ & \cup [E_{k_1} \cap \cdots \cap E_{k_r} \cap E_{j_1}^c \cap \cdots \cap E_{j_s}^c \cap E_{j_{s+1}}^c], \end{aligned}$$

and the right hand side is a disjoint union, so the probability of the left hand side is exactly the sum of the probabilities of the two terms on the right. Thus the induction assumption implies

$$\begin{aligned} P(E_{k_1} \cap \cdots \cap E_{k_r} \cap E_{j_1}^c \cap \cdots \cap E_{j_s}^c \cap E_{j_{s+1}}^c) \\ &= P(E_{k_1}) \cdots P(E_{k_r}) P(E_{j_1}^c) \cdots P(E_{j_s}^c) \\ &\quad - P(E_{k_1}) \cdots P(E_{k_r}) P(E_{j_1}^c) \cdots P(E_{j_s}^c) P(E_{j_{s+1}}) \\ &= P(E_{k_1}) \cdots P(E_{k_r}) P(E_{j_1}^c) \cdots P(E_{j_s}^c) [1 - P(E_{j_{s+1}})], \end{aligned}$$

which is exactly what we wanted to show.

p. 114 #14. The problem is just gamblers ruin with an infinitely rich opponent, i.e., with $N = \infty$, and the answer given is just the $N \to \infty$ limit of formula (4.5) for the gambler's ruin problem. The moral is clear: the casino wins in the end.

Self-test: p. 117 #21. Let E be the event that A gets more heads than B, and let R, S, and T be the events respectively that after n flips by each A has more heads, B has more heads, and they are tied. Conditioning yields

$$P(E) = P(E|R)P(R) + P(E|S)P(S) + P(E|T)P(T).$$

Now $P(E|R) = 1$, since if A is has more heads after n flips by each he must finish with more heads; similarly, $P(E|S) = 0$ and $P(E|T) = 1/2$. On the other hand, $P(S) = P(R)$ because the situation is symmetric between A and B when both have flipped n coins, and from $P(R) + P(S) + P(T) = 1$ we have $P(T) = 1 - 2P(R)$. Thus $P(E) = P(R) + (1 - 2P(R))(1/2) = 1/2$.