SOLUTIONS—ASSIGNMENT 4

Chapter 2. Problems: p. 51 #15. For all of these problems, we must count the number of ways to choose the type of hand in question, then divide by $\binom{52}{5}$ to find the probability.

(a) There are $4 = \binom{4}{1}$ ways to choose the suit and then $\binom{13}{5}$ ways to choose the five cards in the suit. Probability: $\binom{4}{1}\binom{13}{5}/\binom{52}{5}$.

(b) There are $13 = \binom{13}{1}$ ways to choose the rank of the pair, then $\binom{4}{2}$ ways to choose the two cards to make up the pair. There are the $\binom{12}{3}$ ways to choose the remaining three ranks, and $4 = \binom{4}{1}$ ways to choose the card in each rank. Probability: $\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3/\binom{52}{5}$.

(c) There are $\binom{13}{2}$ ways to choose the two ranks for the pairs, then $\binom{4}{2}$ ways to choose the cards in each pair. Finally, we can choose the last card as any of the 44 cards of different rank. Probability: $\binom{13}{2}\binom{4}{2}^2\binom{44}{1}/\binom{52}{5}$.

(d) Choose the rank for the triple (13 choices) then the three cards from this rank ($\binom{4}{3} = 4$ choices). Choose two ranks for the remaining two cards from the 12 ranks remaining, then choose a card in each rank. Probability: $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2/\binom{52}{5}$.

(e) Choose the rank for the four in 13 ways, then choose one of 48 remaining cards. Probability: $13 \cdot 48/\binom{52}{5}$.

p. 51 #16c. There are 6^5 possible outcomes. Moreover, there are $\binom{6}{2}$ choices for the numbers that form pairs, and 4 remaining possibilities for the unpaired number. The number of possible arrangements of the 5 objects, two of which appear twice, is $5!/(2! \cdot 2!)$. Thus the probability is $\binom{6}{2} \cdot 4 \cdot 5!/(2! \cdot 2! \cdot 6^5)$.

p. 52 #18. There are $\binom{52}{2}$ ways to choose two cards from a deck; we assume that all these are equally likely. How many of these choices result in a blackjack? For a blackjack there are 4 way to choose the ace and 16 ways to chose the second card—64 different blackjacks in all. The probability is $\binom{4}{1}\binom{16}{1}/\binom{52}{2} = 64/\binom{52}{2}$.

p. 52 #25: There are 26 pairs $\{(m,n) : 1 \le m,n \le 6\}$ whose sum is *not* 5 or 7, so for n rolls, there are $(26)^{n-1} \cdot 4$ outcomes (out of possible $(36)^n$) that do not have a sum of 5 or 7 on rolls 1, 2, ..., $n-1$ and have sum = 5 on the n^{th} roll. So $P(E_n) = (26)^{n-1} \cdot 4/(36)^n = (13/18)^{n-1}(1/9)$, where E_n is this event. The event

$$E = \{\text{a sum of 5 occurs before a sum of 7}\} = \bigcup_{n=1}^{\infty} E_n\ .$$

Since the events E_n are mutually exclusive for different values of n, we have that $P(E) = \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty}(13/18)^{n-1}(1/9) = \frac{\frac{1}{9}}{1-(13/18)} = \frac{2}{5}$, by the formula for the sum of a geometric series. (This is just the answer you might guess without any calculation. Why?)

p. 52 #28. Since there are 19 balls, there are $\binom{19}{3}$ ways of choosing 3 of them. There are $\binom{5}{3}$ ways of choosing three red balls, $\binom{6}{3}$ ways of choosing three blue balls, and $\binom{8}{3}$ ways of choosing three green balls, so that the probability of getting three the same color is

$$\left[\binom{5}{3} + \binom{6}{3} + \binom{8}{3}\right] \Big/ \binom{19}{3}.$$

To get three different colors we may choose a red ball in 5 ways, a blue ball in 6 ways, and a green ball in 8 ways; the probability of doing this is

$$5 \cdot 6 \cdot 8 \Big/ \binom{19}{3}.$$

When we sample with replacement we have 19 choices at each draw, for a total of 19^3 outcomes in the sample space. (When sampling with replacement we must work with ordered samples!) There are 5^3 ways to get three red balls, etc.., so the first answer becomes $[5^3+6^3+8^3]/19^3$. Now there are still the same number of combinations of three different balls, but we can draw each combination in any of 3! orders, so the probability of three different colors is $3! \cdot 5 \cdot 6 \cdot 8/19^3$.

p. 53 #33. Assume that we capture the 4 elk all at once (sampling without replacement) and that all are equally likely to be captured. This is TE 15, with tagged elk corresponding to white balls and with $M = 5$, $N = 15$, $r = 4$, and $k = 2$. The probability is $\binom{5}{2}\binom{15}{2} \Big/ \binom{20}{4}$.

p. 53 #37: There are $\binom{10}{5} = 252$ possible examinations. (a) Of these, the student can do $\binom{7}{5} = 21$ completely correctly, with probability $21/252 = 7/84 = 1/12 \approx 0.08333\cdots$. (b) She can do $\binom{7}{4}\binom{3}{1} = 35 \cdot 3 = 105$ with four correct, so her chances of getting at least 4 are $\frac{105+21}{252} = \frac{126}{252} = \frac{1}{2} = 0.5$.

p. 53 #42. There are 36 possible outcomes on each throw and thus 36^n outcomes in the sample space of the experiment. On each roll there are 35 outcomes other than double 6, so that the probability of not getting a double 6 at all is $35^n/36^n = (35/36)^n$. The complementary event of at least one double 5 has probability $1 - (35/36)^n$; this expression has value .4914 for $n = 24$ and value .5055 for $n = 25$.

p. 53 #43: The sample space S is the set of all arrangements (permutations) of the N people, and $|S| = N!$. For arranging in a line, we have $N - 1$ choices of a pair of adjacent places for A and B, 2 choices for assigning A and B to these places, and then $(N - 2)!$ choices for arranging the remaining people. The total number of arrangements with A next to B is $= 2 \cdot (N - 1) \cdot (N - 2)!$; so the probability is $2/N$. For arranging in a circle, we have N choices for A, then 2 choices for B (left or right of A), then $(N - 2)!$ choices for the remaining people. So there are $2 \cdot N \cdot (N - 2)!$ choices this time, and the probability is 2/(N-1).

p. 53 #48. Let us cheat and assume that birthdays are equally likely to fall in any month. Then there are 12^{20} equally likely ways that the birthdays could be distributed among the months (each of 20 birthdays gets to choose one of 12 months). To get the outcome described in the problem we must first divide the months: 4 months to have 2 birthdays each, 4 to have 3 birthdays each, and 4 to have no birthdays; then divide the people among the 8 months with birthdays. Each of these division problems gives a multinomial coefficient, so the probabilty is

$$\binom{12}{4\ 4\ 4}\binom{20}{2\ 2\ 2\ 2\ 3\ 3\ 3\ 3} \Big/ 12^{20}.$$

p. 53 #49. This is TE 15 with $M = N = r = 6$ and $k = 3$; the probability is $\binom{6}{3}^2 \Big/ \binom{12}{6}$.

p. 54 #51. The answer is $\frac{\binom{n}{m}(N-1)^{n-m}}{N^n}$. To see this, first choose m balls to go in the first compartment, and then distribute the remaining $n-m$ arbitrarily among the last $N-1$ compartments.

p. 54 #54. The given formula overestimates the probability because hands with two or more voids are counted more than once. To use the inclusion-exclusion formula, let E_k, $k=1,\ldots,4$, be the event of being void in suit k. Clearly $|E_k| = \binom{39}{13}$, $|E_kE_j| = \binom{26}{13}$ for $k \neq j$, $|E_kE_jE_i| = \binom{13}{13} = 1$ for k,j, and i all distinct, and $|E_1E_2E_3E_4| = 0$. Since there are $\binom{4}{1}$ sets E_k, $\binom{4}{2}$ pairs E_kE_j with $k \neq j$, etc., the probability of at least one void is

$$P(E_1 \cup E_2 \cup E_3 \cup E_4) = \left[\binom{4}{1}\binom{39}{13} - \binom{4}{2}\binom{26}{13} + \binom{4}{3}\binom{13}{13}\right] \Bigg/ \binom{52}{13}.$$

Theoretical exercises: p. 55 #15. We assume that we are sampling without replacement. There are $\binom{M+N}{r}$ ways to choose r balls from $M+N$; this is the size of our sample space, and we assume that all choices are equally likely. To get a sample with exactly k white balls we must also choose $r-k$ black balls; there are $\binom{M}{k}$ and $\binom{N}{r-k}$ ways to choose the white and black balls, respectively, and by our basic principle we multiply these to get the total number of choices. The probability is

$$\binom{M}{k}\binom{N}{r-k} \Bigg/ \binom{M+N}{r}.$$

Self-Test: p. 57#6. There are $6 \cdot 10 = 60$ ways to choose the two balls; we assume that they are equally probable. There are $3 \cdot 4$ ways to choose two red balls and $3 \cdot 6$ ways to choose two black balls, a total of $12 + 18 = 30$ ways to choose two balls of the same color. The probability is $30/60 = .5$. (Why is this answer obvious?)