SOLUTIONS—ASSIGNMENT 23

Chapter 8. Problems: p. 424 #3. Let $Y = \overline{X}_n = (\sum_{i=1}^n X_i)/n$ be the class average. Then Y has standard deviation $5/\sqrt{n}$, so an application of the central limit theorem implies

$$P\{|Y - 75| \le 5\} = P\left\{-\frac{5}{5/\sqrt{n}} \le \frac{Y-75}{5/\sqrt{n}} \le \frac{5}{5/\sqrt{n}}\right\} \simeq 2\Phi\big(\sqrt{n}\big) - 1.$$

Requiring that this probability have value at least .9 leads to $\Phi\big(\sqrt{n}\big) \ge .95$, $\sqrt{n} \ge 1.65$, and $n = 3$. Of course, the entire exercise is not worth much: the central limit theorem is unlikely to yield a good approximation with $n = 3$.

p. 424 #4. (a) $E[X_i] = 1$, so $E\left[\sum_{i=1}^{20} X_i\right] = 20 \cdot 1 = 20$ and $P\{\sum X_i > 15\} = P\{\sum X_i \ge 16\} \le 20/16 = 5/4$, a useless bound, since we know that any probability is at most 1.

(b) $E[X_i] = 1$ and $\mathrm{Var}(X_i) = 1$, so if $X = \sum_{i=1}^{20} X_i$ then $E[X] = 20$ and $\mathrm{Var}(X) = 20$. Thus

$$P\{X > 15\} = P\{(X-20)/\sqrt{20} > -5/\sqrt{20}\} \simeq \Phi(5/\sqrt{20}) = .8686.$$

(The exact value is .8435. If the continuity correction (i.e., first writing the quantity to be estimated as $P\{X \ge 15.5\}$) had been used above to get a better approximation, we would have obtained .8438.)

p. 425 #5. Let X_i be the error introduced in rounding the i^{th} number; since X_i is uniform on $[-.5, .5]$, $E[X_i] = 0$, $\mathrm{Var}(X_i) = \sigma^2 = 1/12$. If $X = \sum_{i=1}^{50} X_i$, then we want to estimate $P\{|X| \ge 3\}$. Now by the central limit theorem $X/\sqrt{\mathrm{Var}(X)} = X/\sqrt{n}\sigma$ is approximately standard normal, and

$$P\{|X| \ge 3\} = P\left\{\left|\frac{X}{\sqrt{\mathrm{Var}(X)}}\right| \ge \frac{3}{\sqrt{50/12}}\right\} \simeq 2(1 - \Phi(1.47)) = .1416\ .$$

Theoretical exercises: p. 427 #8. For t an integer we know by Example 3b on page 261 that a gamma random variable Y_t with parameters (t, λ) is the sum of t independent exponential random variables X_i, each with parameter λ. Thus by the central limit theorem, Y_t is approximately normal; more specifically,

$$\tilde{Z}_t = \frac{Y_t - tE[X_i]}{\sqrt{t\,\mathrm{Var}(X_i)}} = \frac{\lambda Y_t - t}{\sqrt{t}}$$

is approximately standard normal. (For noninteger t this argument does not apply, but it seems reasonable to assume that for large t the distribution of $\tilde{Z}_t$ does not vary too much as t varies between successive integers, which would imply that $\tilde{Z}_t$ is approximately standard normal for all large t. The assumption can, in fact, be more or less proven using Proposition 3.1 on page 261.)