SOLUTIONS—ASSIGNMENT 22

Chapter 8. Problems: p. 424 #1. This is a direct application of Chebyshev's inequality, although we have to be a little careful: the quantity that we have to estimate has a strict inequality where Chebyshev has a weak inequality. Still, it all works out. Set $\mu = 20$ and $\sigma^2 = 20$; then

$$P\{0 \le X \le 40\} = P\{-20 \le X - 20 \le 20\} = P\{|X-\mu| \le 20\} = 1 - P\{|X-\mu| > 20\}$$
$$\ge 1 - P\{|X-\mu| \ge 20\} \ge 1 - \frac{\sigma^2}{20^2} = \frac{19}{20}.$$

p. 424 #2. (a) By Markov's inequality, with X a randomly selected student's test score,

$$P\{X > 85\} \le P\{X \ge 85\} \le \frac{E[X]}{85} = \frac{15}{17}.$$

Note that if we assume that the exam scores are integers, then we can do better:

$$P\{X > 85\} = P\{X \ge 86\} \le \frac{E[X]}{86} = \frac{75}{86} < \frac{15}{17}.$$

(Note also how we can *saturate* this inequality: if $P\{X = 86\} = 75/86$ and $P\{X = 0\} = 11/86$ then $E[X] = 0(11/86) + 86(75/86) = 75$ and $P\{X \ge 86\} = 75/86$.)

(b) $P\{|X - 75| \le 10\} = 1 - P\{|X - 75| > 10\} \ge 1 - 25/100 = 3/4$.

(c) Suppose that n students take the test, with scores $X_1, \ldots, X_n$, and let $\overline{X}_n$ be the class average: $\overline{X}_n = \left(\sum_{i=1}^n X_i\right)/n$. Then $E[\overline{X}_n] = 75$ and $\mathrm{Var}(\overline{X}_n) = 25/n$, so

$$P\{|\overline{X}_n - 75| \le 5\} = 1 - P\{|\overline{X}_n - 75| > 5\} \ge 1 - P\{|\overline{X}_n - 75| \ge 5\} \ge 1 - \frac{25/n}{25} = 1 - \frac{1}{n}.$$

To make this probability at most .9 we take $n \ge 10$.

Theoretical exercises: p. 426 #1. This is just a rewriting of Chebyshev's inequality:

$$P\{|X-\mu| \ge k\sigma\} \le \frac{\sigma^2}{(k\sigma)^2} = \frac{1}{k^2}.$$