SOLUTIONS—ASSIGNMENT 21

Chapter 7. Problems: p. 385 #80. From Table 7.1 we see that X is Poisson with parameter 2 and Y is binomial with parameters $(10, 3/4)$. Thus from independence:

(a)

$$\begin{aligned} P\{X+Y=2\} &= P\{X=0,Y=2\}+P\{X=1,Y=1\}+P\{X=2,Y=0\} \\ &= P\{X=0\}P\{Y=2\}+P\{X=1\}P\{Y=1\}+P\{X=2\}P\{Y=0\} \\ &= \frac{e^{-2}}{4^{10}}\left[1\cdot\binom{10}{2}3^2+\frac{2}{1}\cdot\binom{10}{1}3+\frac{4}{2}\cdot\binom{10}{0}\right]=\frac{467}{4^{10}}e^{-2}; \end{aligned}$$

(b)

$$\begin{aligned} P\{XY=0\} &= P(\{X=0\}\cup\{Y=0\}) = P\{X=0\}+P\{Y=0\}-P\{X=Y=0\} \\ &= e^{-2}+\frac{1}{4^{10}}-\frac{e^{-2}}{4^{10}}; \end{aligned}$$

(c)

$$E[XY]=E[X]E[Y]=2\cdot\left[10\cdot\frac{3}{4}\right]=15.$$

Theoretical exercises: p. 389 #46. If X is uniform on $[a,b]$, then

$$M_X(t)=E[e^{tX}]=\frac{1}{b-a}\int_a^b e^{tx}\,dx=\frac{1}{b-a}\left.\frac{e^{tx}}{t}\right|_a^b=\frac{e^{bt}-e^{at}}{t(b-a)}. \qquad (*)$$

Note that $M_X(0)$ is not defined by $(*)$, but that $M_X(0)=E[1]=1$ directly from the definition of M_X, and also that l'Hospital's rule shows that $\lim_{t\to0}M_X(t)=1$.

Now we want to compute $M_X'(0)$. Because $(*)$ does not make sense at $t=0$ we cannot differentiate $(*)$ and then set $t=0$ (in fact, one gets the right answer by differentiating $(*)$ and then taking the $t\to0$ limit using l'Hospital's rule, but technically this requires justification). We can, however, compute M_X' from the definition of derivative:

$$E[X]=M_X'(0)=\lim_{t\to0}\frac{M_X(t)-M_X(0)}{t}=\lim_{t\to0}\frac{e^{bt}-e^{at}-t(b-a)}{t^2(b-a)}=\frac{a+b}{2},$$

where we have used l'Hospital's rule. Of course, this is a familiar answer. We can compute $E[X^2]=M_X''(0)=\lim_{t\to0}[M_X'(t)-M_X'(0)]/t]=(b^2+ab+a^2)/3$ similarly.

p. 390 #51.

$$\Psi''(t) = \frac{d^2}{dt^2} \log M(t) = \frac{d}{dt} \frac{M'(t)}{M(t)} = \frac{M''(t)M(t) - M'(t)M'(t)}{M(t)^2}.$$

Setting $t = 0$ and using $M(0) = 1$, $M'(0) = E[X]$, and $M''(0) = E[X^2]$ gives $\Psi''(0) = E[X^2] - E[X]^2 = \mathrm{Var}(X)$.

p. 390 #52. . Let $X = \sum_{i=1}^n X_i$. From Table 7.2, $M_{X_i}(t) = \lambda/(\lambda - t)$ for all $t < \lambda$. Since the X_i are independent,

$$M_X(t) = M_{X_1+\cdots+X_n}(t) = M_{X_1}(t) \cdots M_{X_n}(t) = \left(\frac{\lambda}{\lambda - t}\right)^n.$$

According to Table 7.2 this is just the moment generating function of a random variable which has a gamma distribution with parameters (n, λ); since the moment generating function determines the distribution, X must have this same distribution.