SOLUTIONS—ASSIGNMENT 19

Chapter 7. Problems: p. 381 #33. a) $E[X^2] = \text{Var}(X) + E[X]^2 = 6$, so $E[(2+X)^2] = E[4+4X+X^2] = 4+4+6 = 14$.

b) A constant random variable is independent of any other and has variance zero, so $\text{Var}(4+3X) = \text{Var}(4) + \text{Var}(3X) = 9\,\text{Var}(X) = 45$.

p. 381 #34. b) Let X be the number of wives seated next to their husbands. We can write X as a sum of Bernoulli random variables in various ways; some are more convenient than others for the computation of variance. We take $X = \sum_{i=1}^{20} X_i$, where $X_i = 1$ if there is a married couple seated at place i and the next place, that is, at i and $i+1$ for $i = 1, \ldots, 19$, and at i and 1 if $i = 20$ (for convenience, we write i and $i+1$ in this case also, taking $20 + 1 = 1$).

Now $\text{Var}(X) = \sum_{i=1}^{20} \text{Var}(X_i) + 2\sum_{1\le i<j\le 20} \text{Cov}(X_i, X_j)$. Since X_i is Bernoulli, we have $\text{Var}(X_i) = (1/19)[1-(1/19)] = 18/361$. To compute $\text{Cov}(X_i, X_j) = E[X_iX_j] - E[X_i]E[X_j]$ we note that X_iX_j is Bernoulli, taking value 1 if *both* places i and $i+1$ *and* places j and $j+1$ are occupied by married couples. If i and j are adjacent (this covers 20 choices of i and j) this is impossible, so $E[X_iX_j] = 0$; otherwise ($\binom{20}{2} - 20 = 170$ choices) there are (20)(19)(18)(17) ways to seat people at these four seats, of which (20)(18) yield two couples in the right places, so $E[X_iX_j] = 1/(19 \cdot 17) = 1/323$. Thus

$$\text{Var}(X) = \sum_{i=1}^{20} \text{Var}(X_i) \; + 2\left(\sum_{1\le i<j\le 20} E[X_iX_j] \; - \sum_{1\le i<j\le 20} E[X_i]E[X_j]\right)$$

$$= 20\frac{18}{361} + 2\left(20\cdot 0 + 170\frac{1}{323} - 190\left(\frac{1}{19}\right)^2\right) = \frac{360}{361}.$$

p. 381 #36. Write $X = \sum_{i=1}^{n} X_i$ and $Y = \sum_{i=1}^{n} Y_i$, where X_i (respectively Y_i) is a Bernoulli random variable with $X_i = 1$ if a 1 occurs (respectively $Y_i = 1$ if a 2 occurs) on the i^{th} roll, and $X_i = 0$ (respectively $Y_i = 0$) otherwise. Clearly $E[X_i] = E[Y_i] = 1/6$. Now X_iY_i is always 0, so $\text{Cov}(X_i, Y_i) = E[X_iY_i] - E[X_i]E[Y_i] = -1/36$, but if $i \ne j$ then X_i and Y_j are independent, and $\text{Cov}(X_i, Y_j) = 0$. Thus

$$\text{Cov}(X, Y) = \sum_{i=1}^{n}\sum_{j=1}^{n} \text{Cov}(X_i, Y_j) = \sum_{i=1}^{n} \text{Cov}(X_i, Y_i) = -\frac{n}{36}.$$

p. 381 #40.

$$E[X] = \iint xf(x,y)\,dx\,dy = 2\int_0^\infty \int_0^x e^{-2x}\,dy\,dx = 2\int_0^\infty xe^{-2x}\,dx = \frac{1}{2};$$

$$E[Y] = \iint yf(x,y)\,dx\,dy = 2\int_0^\infty \int_0^x y\frac{e^{-2x}}{x}\,dy\,dx = 2\int_0^\infty \frac{x^2}{2}\frac{e^{-2x}}{x}\,dx = \frac{1}{4};$$

$$E[XY] = \iint xyf(x,y)\,dx\,dy = 2\int_0^\infty \int_0^x ye^{-2x}\,dy\,dx = 2\int_0^\infty \frac{x^2}{2}e^{-2x}\,dx = \frac{1}{4}.$$

So $\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = 1/8$.

p. 381 #44. a) Think of placing the pairs in some distinguishable order. If X is the number of mixed-sex pairs then $X = \sum_{i=1}^{10} X_i$, where X_i is Bernoulli with $X_i = 1$ if and only if the i^{th} pair is mixed-sex. Clearly

$$P\{X_i = 1\} = \frac{100}{\binom{20}{2}} = \frac{10}{19},$$

(why is this obvious?) and

$$P\{X_iX_j = 1\} = \frac{100\cdot 81}{\binom{20}{2}\binom{18}{2}} = \frac{90}{323}, \text{ if } i \neq j.$$

Thus $E[X] = \sum_1^{10} E[X_i] = 100/19$. For $i \neq j$, $\mathrm{Cov}(X_i, X_j) = 90/323 - (10/19)^2 = 10/6137$, and

$$\mathrm{Var}(X) = \sum_1^{10} \mathrm{Var}(X_i) + 2\sum_{1\le i<j\le 10} \mathrm{Cov}(X_i,X_j) = 10\frac{10}{19}\frac{9}{19} + 90\frac{10}{6137} = \frac{16200}{6137}.$$

b) The analysis is almost the same, but now we let $X_i = 1$ if the i^{th} pair is a married couple. Now

$$P\{X_i = 1\} = \frac{10}{\binom{20}{2}} = \frac{1}{19}, \qquad P\{X_iX_j = 1\} = \frac{10\cdot 9}{\binom{20}{2}\binom{18}{2}} = \frac{1}{323}, \text{ if } i \neq j.$$

So $E[X] = 10/19$, $\mathrm{Cov}(X_i, X_j) = 2/6137$ for $i \neq j$, and

$$\mathrm{Var}(X) = 10\frac{1}{19}\frac{18}{19} + 90\frac{2}{6137} = \frac{3240}{6137}.$$