SOLUTIONS—ASSIGNMENT 18

Chapter 7. Problems: p. 378 #4.(a) $E(XY) = \int_0^1 \int_0^y xyy^{-1}\,dxdy = \int_0^1 \int_0^y x\,dxdy = \frac{1}{2}\int_0^1 y^2\,dy = 1/6$. (b) $E(X) = \int_0^1 \int_0^y xy^{-1}\,dxdy = 1/4$. (c) $E(Y) = \int_0^1 \int_0^y yy^{-1}\,dxdy = 1/2$.

p. 379 #14. Let X be the number of stages needed. Then $X = \sum_{i=1}^m X_i$, where X_i is geometric with parameter $1-p$. (Why?) The answer given follows immediately.

p. 379 #19. (a) Let Y be the total number of catches until the first type 1 catch; Y is geometric with parameter $p = P_1$, the success probability on each trial. Thus $E[Y] = 1/P_1$. If X is the number of catches *before* the first type 1 catch, then $X = Y - 1$ and $E[X] = E[Y-1] = E[Y] - 1 = (1-P_1)/P_1$.

(b). Let Z be the number of types caught before the first type 1 catch. Then $Z = \sum_{j=2}^r Z_j$, where for each type $j \neq 1$, Z_j is a Bernoulli random variable with $Z_j = 1$ if an insect of type j is caught before the first type 1 catch, and $Z_j = 0$ otherwise. Since Z_j is Bernoulli, $E[Z_j] = P\{Z_j = 1\}$, and $P\{Z_j = 1\} = P_j/(P_1 + P_j)$ (this formula is an old friend—see **Example 4h** in Chapter 3). Thus

$$E[Z] = \sum_{j=2}^r E[Z_j] = \sum_{j=2}^r \frac{P_j}{P_1 + P_j}.$$

p. 380 #21. Assume as usual that there are 365 days in the year, all equally likely as birthdays. Then the number of people Z_i having birthdays on a given day i is binomial $(100, 1/365)$.

(a) For each day i, let $X_i = 1$ if exactly three people have birthdays on day i and let $X_i = 0$ otherwise; then

$$E\left[\sum_{i=1}^{365} X_i\right] = \sum_{i=1}^{365} P\{X_i = 1\} = \sum_{i=1}^{365} P\{Z_i = 3\} = 365\binom{100}{3}\left(\frac{1}{365}\right)^3\left(\frac{364}{365}\right)^{97} = .9301.$$

(b) This is similar: let $Y_i = 1$ if at least one birthday occurs on day i, $Y_i = 0$ otherwise. Then

$$E\left[\sum_{i=1}^{365} Y_i\right] = \sum_{i=1}^{365} P\{Y_i = 1\} = \sum_{i=1}^{365} P\{Z_i > 0\} = 365\left[1 - \left(\frac{364}{365}\right)^{100}\right] = 87.58.$$

p. 380 #23. Following the hint, we need only compute $E[X_i] = P\{X_i = 1\}$ and $E[Y_i] = P\{Y_i = 1\}$.

Now in selecting n balls from an urn containing N balls, the probability that any particular ball is selected is n/N. (Why is this obvious?) Thus the probability that a particular ball is moved from urn 1 to urn 2 is $2/11$, and the probability that a particular ball which is in urn 2 at the beginning of the second stage is selected for the trio is $3/20$. Thus $P\{Y_i = 1\} = 3/20$ and $P\{X_i = 1\} = (2/11)(3/20) = 3/110$. Finally, $E[\sum X_i + \sum Y_i] = 5(3/110) + 8(3/20) = 147/110$.

p. 381 #34. a) Let X be the number of wives seated next to their husbands. We can write X as a sum of Bernoulli random variables in various ways; some are more convenient than others for the computation of variance (34b). We take $X = \sum_{i=1}^{20} X_i$, where $X_i = 1$ if there is a married couple seated at place i and the next place, that is, at i and $i+1$ for $i = 1, \ldots, 19$, and at i and 1 if $i = 20$ (for convenience, we write i and $i+1$ in this case also, taking $20 + 1 = 1$).

Now there are $\binom{20}{2}$ ways to choose two people to sit at two specified chairs, and $\binom{10}{1}$ ways to choose a married couple, so $E[X_i] = P\{X_i = 1\} = \binom{10}{1}\Big/\binom{20}{2} = 1/19$. (Why is this obvious?) Thus $E[X] = 20(1/19) = 20/19$.

Theoretical exercises: p. 385 #1. Let $f(a) = E[(X-a)^2]$. We want to show that $f(a)$ has a strict minimum at $a = E[X]$, that is, that $f(a) \geq f(E[X])$ for all a, with equality holding only for $a = E[X]$. It helps to write

$$f(a) = E[(X-a)^2] = E[X^2 - 2aX - a^2] = E[X^2] - 2aE[X] + a^2. \qquad (*)$$

Method 1: Just compute, from $(*)$,

$$f(a) - f(E[X]) = \big(E[X^2] - 2aE[X] + a^2\big) - \big(E[X^2] - 2E[X]^2 + E[X]^2\big) = (a - E[X])^2 \geq 0$$

and note that equality holds—that is, $(a - E[X])^2 = 0$—only if $a = E[X]$.

Method 2: From $(*)$, $f(a)$ is just a quadratic polynomial with $f'(a) = 2(a - E[X])$ and $f''(a) = 2$. Since f'' is strictly positive, f has an absolute minimum where $f' = 0$—that is, at $a = E[X]$.