SOLUTIONS—ASSIGNMENT 17

Chapter 6. Problems: p. 294 #42. For $X$ chosen at random from $\{1,2,3,4,5\}$ and then $Y$ chosen at random from $\{1,\dots,X\}$:

**(a)** $p(x,y) = P\{X = x, Y = y\} = P\{Y = y|X = x\}P\{X = x\} = (1/x)(1/5) = 1/(5x)$ for $1 \le y \le x \le 5$, and is 0 otherwise ($x, y$ integers).

**(b)** The marginal pmf $p_Y(y)$ for $Y$ is $P\{Y = y\} = \sum_{x=y}^{x=5} \frac{1}{5x}$. This has values 137/300, 77/300, 47/300, 27/300, 12/300 for $Y = 1,2,3,4,5$. The conditional pmf $p_{X|Y}(x|y)$ for $X$ given $Y$ is $P\{X = x, Y = y\}/P\{Y = y\}$. Values: $y = 1$: 60/137, 30/137, 20/137, 15/137, 12/137 for $x = 1,2,3,4,5$. $y = 2:$ 30/77, 20/77, 15/77, 12/77 for $x = 2,3,4,5$. $y = 3$: 20/47, 15/47, 12/47 for $x = 3,4,5$. $y = 4:$ 15/27, 12/27 for $x = 4,5$. $y = 5$: 1 for $x = 5$.

**(c)** $X$ and $Y$ are clearly not independent, since the possible values of $Y$ depned upon $X$. For example, $P\{X = 1, Y = 2\} = 0$ but $P\{X = 1\}P\{Y = 2\} = (1/5)(77/300) \neq 0$.

p. 295 #44. a) $p_{X|Y}(x|1)$: 1/2, 1/2. $p_{X|Y}(x|2)$: 1/3, 2/3. b) No! c) 1/2, 7/8, 1/8.

p. 295 #46. The conditional density $f_{Y|X}(y|x)$ is defined for $x > 0$ and is given there by

$$f_{Y|X}(y|x) = \frac{c(x^2-y^2)e^{-x}}{\int_{-x}^{x} c(x^2-y^2)e^{-x}\,dy} = \frac{(x^2-y^2)}{\int_{-x}^{x}(x^2-y^2)\,dy} = \frac{3}{4}\left(\frac{1}{x} - \frac{y^2}{x^3}\right)$$

for $-x < y < x$. Note that the answer in the back is the corresponding cdf.

Theoretical exercises: p. 296 #9. Let $Y = \min(X_1,\dots,X_n)$ be the minimum of the random variables $X_1$, $X_2$, …, $X_n$. $Y$ is greater than some number $y$ if and only if all $X_i$ are greater than $y$; and for any $i$, $P\{X_i > y\} = e^{-\lambda y}$. Thus, using the independence of the $X_i$,

$$\begin{aligned} F_Y(y) &= P\{Y \le y\} = 1 - P\{Y > y\} = 1 - P\{X_1 > y,\dots,X_n > y\} \\ &= 1 - P\{X_1 > y\}\cdots P\{X_n > y\} = 1 - \left(e^{-\lambda y}\right)^n = 1 - e^{-n\lambda y}. \end{aligned}$$

$Y$ is exponential with parameter $n\lambda$.