SOLUTIONS—ASSIGNMENT 15

Chapter 6. Problems: p. 291 #1. **(a)** See the table.

Table for p. 291 #1(a)

$X\backslash Y$	2	3	4	5	6	7	8	9	10	11	12
1	1/36										
2		1/18	1/36								
3			1/18	1/18	1/36						
4				1/18	1/18	1/18	1/36				
5					1/18	1/18	1/18	1/18	1/36		
6						1/18	1/18	1/18	1/18	1/18	1/36

p. 291 #1. **(b)** See the table.

Table for p. 291 #1(b)

$Y\backslash X$	1	2	3	4	5	6
1	1/36					
2	1/36	2/36				
3	1/36	1/36	3/36			
4	1/36	1/36	1/36	4/36		
5	1/36	1/36	1/36	1/36	5/36	
6	1/36	1/36	1/36	1/36	1/36	6/36

p. 291 #1. **(c)** See the table.

Detailed solution: Sample space $S = \{(m,n) : m,n \text{ integers } 1,2,\ldots,6\ \}$, $|S| = 36$.

(a) $X(m,n) = \max\{m,n\}$ and $Y(m,n) = m+n$. The joint range of (X,Y) is all pairs of integers (x,y) with $x+1 \le y \le 2x$ and $1 \le x \le 6$, $2 \le y \le 12$. If $y = 2x$, then only one point $(x,x) \in S$ gives this value, so $p(x,2x) = 1/36$. If $x < y < 2x$ then the points $(x, y-x)$ and $(y-x, x)$ in S both give this value for (X,Y). So $p(x,y) = 2/36$ in this case.

(b) Now $X(m,n) = m$ and $Y(m,n) = \max\{m,n\}$. The joint range of (X,Y) is all pairs of integers (x,y) with $x \le y$ and $x,y = 1,\ldots,6$. For $x = y$, the points $(x,1),(x,2),\ldots,(x,x)$ of S give this value for (X,Y), so $p(x,x) = x/36$. For $x < y$, the only point of S giving this value is (x,y). So $p(x,y) = 1/36$.

Table for p. 291 #1(c)

$Y\backslash X$	1	2	3	4	5	6
...	...	...	...	...	...	...
1	1/36					
2	2/36	1/36				
3	2/36	2/36	1/36			
4	2/36	2/36	2/36	1/36		
5	2/36	2/36	2/36	2/36	1/36	
6	2/36	2/36	2/36	2/36	2/36	1/36

(c) Now $X(m,n) = \min\{m,n\}$ and $Y(m,n) = \max\{m,n\}$. The joint range of (X,Y) is the same as in (b). If $x = y$, then the only point of S giving this value for (X,Y) is (x,y). So $p(x,y) = 1/36$. If $x < y$ then the points (x,y) and (y,x) of S both give this value of (X,Y).

p. 292 #7. If $X_1 = i$ and $X_2 = j$ then we know exactly what happened on the first $i+j+2$ trials: i failures, then a success, then j failures, then another success. The probability of this is $(1-p)^i p(1-p)^j p = (1-p)^{i+j}p^2$.

p. 292 #8. (a)

$$1 = \iint f(x,y)dxdy = c\int_0^\infty \int_{-y}^{y} (y^2 - x^2)e^{-y}\,dx\,dy$$
$$= c\int_0^\infty e^{-y}\Big[y^2x - x^3/3\Big]_{-y}^{y} dy = \frac{4c}{3}\int_0^\infty y^3 e^{-y} dy = 8c.$$

Thus $c = 1/8$.

(b) Since $Y > 0$, $f_Y(y) = 0$ for $y < 0$. For $y > 0$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{1}{8}\int_{-y}^{y} (y^2 - x^2)e^{-y}\,dx = \frac{1}{6}y^3 e^{-y}.$$

For any x, $f(x,y) = 0$ unless $y > |x|$. So

$$f_X(x) = \frac{1}{8}\int_{|x|}^{\infty} (y^2 - x^2)e^{-y}\,dy = \frac{1}{8}\Big[(-y^2 - 2y - 2 + x^2)e^{-y}\Big]_{|x|}^{\infty} = \frac{1}{4}(|x|+1)e^{-|x|}.$$

(c) By symmetry, $E(X) = 0$, since $f_X(x)$ is even.

p. 292 #9. (a) $f(x,y)$ is clearly non-negative. The other requirement for a density is that it be normalized:

$$\frac{6}{7}\int_0^1 \int_0^2 (x^2 + xy/2)\,dy\,dx = \frac{6}{7}\int_0^1 (2x^2 + x)\,dx = \frac{6}{7}\left(\frac{2}{3} + \frac{1}{2}\right) = 1.$$

(b) $f_X(x) = \frac{6}{7}\int_0^2 (x^2 + xy/2)\,dy = \frac{6}{7}(2x^2 + x),\ 0 < x < 1.$

(c)

$$P\{X > Y\} = \frac{6}{7}\int_0^1 \int_0^x (x^2 + xy/2)\,dy\,dx = \frac{6}{7}\int_0^1 \frac{5x^3}{4}\,dx = \frac{15}{56}.$$

(d)

$$P\{X < \frac{1}{2}\} = \int_0^{1/2} f_X(x) = \frac{6}{7}\Big[\frac{2x^3}{3} + \frac{x^2}{2}\Big]_0^{1/2} = \frac{5}{28}.$$

So

$$\begin{aligned} P\Big(\{Y > \frac{1}{2}\}\Big|\{X < \frac{1}{2}\}\Big) &= \frac{28}{5}P\Big(\{Y > \frac{1}{2}\} \text{ and } \{X < \frac{1}{2}\}\Big) \\ &= \frac{28}{5}\frac{6}{7}\int_0^{1/2}\int_{1/2}^2 (x^2 + \frac{xy}{2})\,dy\,dx \\ &= \frac{24}{5}\int_0^{1/2}\left(\frac{3x^2}{2} + \frac{15x}{16}\right)\,dx \\ &= \frac{24}{5}\left(\frac{1}{16} + \frac{15}{128}\right) = \frac{69}{80}. \end{aligned}$$

(e) Using the result of part (b), we have that $E(X) = \int_0^1 x f_X(x)\,dx = \frac{6}{7}\int_0^1 (2x^3 + x^2)\,dx = 5/7$.

(f) $f_Y(y) = \int_0^1 f(x,y)\,dx = \frac{1}{14}(4 + 3y),\ 0 < y < 2$. Thus $E(Y) = \int_0^2 y f_Y(y)\,dy = 8/7$.

p. 292 #10. **(a)** $P\{X < Y\} = \displaystyle\int_0^\infty \int_0^y e^{-(x+y)}\,dx\,dy = \int_0^\infty e^{-y}[1 - e^{-y}]\,dy = 1 - 1/2 = 1/2$. (This could be seen without calculation by the symmetry in X and Y). **(b)** Since $f(x,y) = e^{-x}\cdot e^{-y}$, the marginal density of X is exponential with parameter $\lambda = 1$. Thus $P\{X < a\} = 1 - e^{-a}$, which can also be easily seen by integrating $f(x,y)$ over the region $0 < x < a,\ 0 < y < \infty$.