SOLUTIONS—ASSIGNMENT 13

Chapter 5. Problems: p. 229 #15. Here $Z = (X - 10)/6$ is standard normal. So, using linear interpolation to read the Φ table for arguments with three decimal places and recalling that $\Phi(x) = 1 - \Phi(-x)$, we find

(a) $P\{X > 5\} = P\{Z > -5/6\} = 1 - \Phi(-5/6) = \Phi(5/6) = .7976$;

(b) $P\{4 < X < 16\} = P\{-1 < Z < 1\} = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = .6826$;

(c) $P\{X < 8\} = P\{Z < -2/6\} = \Phi(-1/3) = 1 - \Phi(1/3) = .3694$;

(d) $P\{X < 20\} = P\{Z < 5/3\} = \Phi(5/3) = .9522$;

(e) $P\{X > 16\} = P\{Z > 1\} = 1 - \Phi(1) = .1587$.

p. 229 #16. Define $E = \{\text{rainfall } \leq 50'' \text{ in one year}\}$. Then $P\{E\} = P\{X \leq 50\}$ where X is normal with $\mu = 40$ and $\sigma = 4$. Set $Z = (X - 40)/4$. Z is standard normal, so $P\{X \leq 50\} = P\{(X - 40)/4 \leq (50 - 40)/4\} = P\{Z \leq 2.5\} = \Phi(2.5) \approx .9938$. Assume total rainfall amounts in successive years are mutually independent events. Then $P\{10 \text{ years with rainfall } \leq 50''\} = (.9938)^{10} = .9397$.

p. 229 #17. Hint: What is μ? What is σ? (For what value of z does $\Phi(z) = .25$?)

p. 230 #23. (a) The number of 6's rolled, X, is binomial $(1000, 1/6)$, so that $\mu = np = (1000/6)$ and $\sigma = \sqrt{np(1-p)} = \sqrt{5000/36}$. Thus $\tilde{Z} = (X - \mu)/\sigma = \big(X - (1000/6)\big)/\sqrt{5000/36}$ is approximately standard normal. Thus

$$\begin{aligned} P\{150 \leq X \leq 200\} &= P\{149.5 \leq X \leq 200.5\} \\ &= P\{-1.46 \leq \tilde{Z} \leq 2.87\} \\ &\simeq \Phi(2.87) - [1 - \Phi(1.46)] = .9258. \end{aligned}$$

(b) Once we know that 6 has appeared 200 times, the number of times 5 appears, Y, is binomial $(800, 1/5)$ (this is pretty clear, but you can check it by explicitly computing $P(\{Y = k\}|\{X = 200\})$), so $\hat{Z} = (Y - 160)/8\sqrt{2}$ is standard normal, and

$$\begin{aligned} P(\{Y < 150\}|\{X = 200\}) &= P\{\hat{Z} \leq (149.5 - 160)/8\sqrt{2}\} \\ &= P\{\hat{Z} \leq -.928\} \simeq 1 - \Phi(.928) = .1762. \end{aligned}$$

p. 230 #26. Let X be the number of heads. If the coin is fair, then $\mu = np = 500$, $\sigma = \sqrt{np(1-p)} = \sqrt{250}$, and

$$P\{X \geq 525\} = P\{X \geq 524.5\} \simeq 1 - \Phi(24.5/\sqrt{250}) = .0606.$$

If the coin is biased, $\mu = np = 550$ and $\sigma = \sqrt{np(1-p)} = \sqrt{247.5}$, and

$$P\{X < 525\} = P\{X < 524.5\} \simeq \Phi(-25.5/\sqrt{247.5}) = .0526.$$

p. 230 #27. Let X be the number of heads. If the coin is fair, then X is binomial with $n = 10,000$ and $p = 1/2$, so that $\mu = np = 5000$ and $\sigma = \sqrt{np(1-p)} = 50$. Thus $Z = (X - 5000)/50$ is approximately standard normal and

$$P\{X \geq 5800\} = P\{Z \geq 800/50 = 16\} \simeq 1 - \Phi(16),$$

which is so small it is "off the chart." It is thus not at all likely that the coin is fair!

p. 230 #32. (a) $P\{T > 2\} = e^{-2\lambda} = .368$. (b) Since T is memoryless, $P(\{T > 10\}|\{T > 9\}) = P\{T > 1\} = e^{-1/2} = .607$.

p. 230 #34. Let T be the lifetime. Then if T is exponential, with parameter 1/20, we have $P(\{T > 30\}|\{T > 10\}) = P\{T > 20\} = e^{-1}$. On the other hand, if T is uniform on $[0, 40]$, then $P(\{T > 30\}|\{T > 10\}) = P\{T > 30\}/P\{T > 10\} = (1/4)/(3/4) = 1/3$.

Chapter 5. Theoretical exercises: p. 232 #13. (b) If X is normal with parameters (μ, σ), then $P\{X \le \mu\} = 1/2$ by symmetry (or $P\{X \le \mu\} = P\{(X-\mu)/\sigma \le 0\} = \Phi(0) = 1/2$ since $Z = (X-\mu)/\sigma$ is standard normal), so the median m is μ.
(c) If X is exponentially distributed then the median m must satisfy

$$\int_m^\infty \lambda e^{-\lambda x}\,dx = 1/2$$

or $e^{-\lambda m} = 1/2$, so $m = \ln 2/\lambda$.

p. 232 #15. Let $Y = cX$. Then $F_Y(y) = P\{Y \le y\} = P\{X \le y/c\} = F_X(y/c) = 1 - e^{-\lambda y/c}$ for $y \ge 0$ (and is 0 otherwise), since

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - e^{-\lambda x}, & \text{if } x \ge 0. \end{cases}$$

Thus Y has density

$$f_Y(y) = F_Y'(y) = \begin{cases} 0, & \text{if } y < 0, \\ \frac{\lambda}{c} e^{-\lambda y/c}, & \text{if } y \ge 0. \end{cases}$$

This is the density of an exponential random variable with paramenter λ/c.