SOLUTIONS—ASSIGNMENT 12

Chapter 4. Problems: p. 228 #1. (a) $1 = \int_{-1}^{1} c(1-x^2)\,dx = 4c/3$, so $c = 3/4$.

(b) $F(x) = P\{X \le x\} = \begin{cases} 0, & \text{if } x < -1, \\ \int_{-1}^{x} f(x)\,dx = \dfrac{3x}{4} - \dfrac{x^3}{4} + \dfrac{1}{2}, & \text{if } -1 \le x \le 1, \\ 1, & \text{if } x > 1 \end{cases}.$

p. 228 #2. First we find C:

$$1 = \int_0^\infty f(x)\,dx = C\int_0^\infty xe^{-x/2}\,dx = 4C,$$

so $C = 1/4$. Then

$$P\{X \ge 5\} = \frac{1}{4}\int_5^\infty xe^{-x/2}\,dx = \frac{7}{2}e^{-5/2}.$$

p. 228 #4. (a) $P\{X \ge 20\} = 10\displaystyle\int_{20}^\infty \frac{dx}{x^2} = 10\left[-\frac{1}{x}\right]_{20}^\infty = \frac{1}{2}.$

(b) $F(x) = P\{X \le x\} = \begin{cases} 0, & \text{if } x < 10, \\ 10\int_{10}^{x} x^{-2}\,dx = 1 - \dfrac{10}{x}, & \text{if } x \ge 10. \end{cases}$

(c) Assume that the devices fail or succeed independently. The probability that any given device functions for at least 15 hours is $10\displaystyle\int_{15}^\infty \frac{dx}{x^2} = \frac{2}{3}$. The number Y of the six devices which function for at least 15 hours is thus binomial $(6, 2/3)$, so

$$P\{Y \ge 3\} = \sum_{k=3}^{6}\binom{6}{k}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{6-k} = .900.$$

p. 228 #5. Let c denote the capacity of the tank and X the volume of sales in a week, both measured in thousands of gallons. Then (assuming that $0 \le c \le 1$), $P\{X > c\} = 5\int_c^1 (1-x)^4\,dx = -(1-x)^5\Big|_c^1 = (1-c)^5$, so that $P(\text{supply exhausted}) = P\{X > c\} = 0.01$ when $c = 1 - (0.01)^{1/5} = .6019$. If the station owner builds a 602 gallon tank rather than a 1000 gallon tank he will run out of gas about once every two years.

p. 229 #6. (a)$E[X] = \int_0^\infty xf(x)\,dx = \frac{1}{4}\int_0^\infty x^2e^{-x/2}\,dx = 4$.

(b) $E[X] = c\int_{-1}^{1} x(1-x^2)\,dx = 0$ for any c, since the integrand is an odd function. (Here $c = \frac{3}{4}$ from $1 = \int_{-1}^{1} c(1-x^2)\,dx$, but we don't need this).

(c) $E[X] = \int_5^\infty x(5/x^2)\,dx = 5\log x\Big|_5^\infty = \infty$.

p. 229 #8. $E[X] = \int_0^\infty x\cdot xe^{-x}\,dx = \int_0^\infty x^2e^{-x}\,dx = \Big[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\Big]_0^\infty = 2$

p. 229 #10. The passenger goes to destination A if he/she arrives after the train to B departs and before the train to A arrives. In 7:00–8:00 there are 4 time intervals in which this will happen: 7:05–7:15, 7:20–7:30, 7:35–7:45, and 7:50–8:00. These intervals make up 40/60 of the one-hour period, so if passenger arrival times are uniformly distributed, then $P\{\text{passenger goes to } A\} = 2/3$. The same analysis applies to the period 7:10–8:10.

p. 229 #11. Let X be the distance of the chosen point from the left end of the segment. "Chosen at random" just means that X is uniformly distributed on $[0, L]$. The segments have length X and $L - X$, so the problem asks for the probability that $X/(L - X) < 1/4$, i.e., $X < L/5$, or that $(L - X)/X < 1/4$, i.e., $X > 4L/5$. Each event has probability 1/5, so the probability we want is 2/5.

p. 229 #13. Let X be the number of minutes after 10:00 that the bus arrives; X is uniformly distributed on $[0, 30]$. Then

(a) $P\{X > 10\} = P\{10 < X < 30\} = 20/30 = 2/3$;

(b) $P(\{X > 25\}|\{X > 15\}) = P\{X > 25\}/P\{X > 15\} = (5/30)\big/(15/30) = 1/3$.

p. 229 #14. By Proposition 2.1, $E[X^n] = \int x^n f_X(x)\,dx = \int_0^1 x^n\,dx = 1/(n+1)$. To use the definition we need to compute the density for X^n. Now $F_{X^n}(y) = P\{X^n \le y\} = P\{X \le y^{1/n}\} = y^{1/n}$, so $f_Y(y) = F_Y'(y) = (1/n)y^{(1/n)-1}$. Finally,

$$E[Y] = \int_0^1 y f_Y(y)\,dy = \int_0^1 y\frac{y^{(1/n)-1}}{n}\,dy = \frac{1}{n}\int_0^1 y^{1/n}\,dy = \frac{1}{n}\frac{n}{n+1} = \frac{1}{n+1}.$$

Chapter 5. Theoretical exercises: p. 232 #13. The median m of a continuous random variable X is characterized by the relations $P\{X < m\} = P\{X > m\} = 1/2$. (Note that since X is continuous, $P\{X = m\} = 0$, so we can also say that $P\{X \le m\} = P\{X \ge m\} = 1/2$.)

(a) Clearly if X is uniform on $[a, b]$ then it has probability 1/2 of lying in the interval $[a, (a+b)/2]$ and of lying in $[(a+b)/2, b]$. The median is the midpoint of the interval: $(a+b)/2$.

For Problem 4: now the median z satisfies

$$\frac{1}{2} = P\{X > z\} = 10\int_z^\infty \frac{dx}{x^2} = \frac{10}{z},$$

so $z = 20$; this is just part (a) of problem 4.