SOLUTIONS—ASSIGNMENT 10

Chapter 4. Problems: p. 178 #41. The number X of correct answers is a binomial random variable with parameters $(5, 1/3)$, since the probability of a correct guess on any question is 1/3. Thus

$$P\{X \geq 4\} = \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) + \binom{5}{5}\left(\frac{1}{3}\right)^5 = \frac{11}{243}.$$

p. 178 #42. The number X of correct guesses is a binomial random variable with parameters $n = 10,\ p = 1/2$. Thus the probability that the man makes at least 7 correct guesses is

$$P\{X \geq 7\} = \left[\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right]\left(\frac{1}{2}\right)^{10} = \frac{176}{1024} = \frac{11}{64} = .172\ .$$

p. 179 #44. We want to transmit a digit d, where d is 0 or 1, and to get the message across we transmit *ddddd*. The receiver decodes our message as a d if a majority of the digits received are d—in other words, if not more than 2 of the 5 transmitted digits are corrupted. If we assume that each digit is independently received correctly or incorrectly, then the number X of digits received incorrectly is a binomial RV with parameters $(5, .2)$. Thus the probability that the message will be wrongly decoded is:

$$P\{X \geq 3\} = \binom{5}{3}(.2)^3(.8)^2 + \binom{5}{4}(.2)^4(.8) + \binom{5}{5}(.2)^5) = .058.$$

p. 179 #45. Let X be the number of components which function. Then if R is the event "rain,"

$$\begin{aligned} P\{X \geq k\} &= P(\{X \geq k\}|R)P(R) + P(\{X \geq k\}|R^c)P(R^c) \\ &= \alpha \sum_{j \geq k}\binom{n}{j}p_1^j(1-p_1)^{n-j} + (1-\alpha)\sum_{j \geq k}\binom{n}{j}p_2^j(1-p_2)^{n-j}. \end{aligned}$$

p. 179 #46. This is a lot like the previous problem. Let E is the event that the student is "on" and let X be the number of examiners who pass her. If there are n examiners then she passes if $X > n/2$, and

$$\begin{aligned} P\{X > n/2\} &= P(\{X > n/2\}|E)P(E) + P(\{X > n/2\}|E^c)P(E^c) \\ &= (1/3)\sum_{j > n/2}\binom{n}{j}(.8)^j(.2)^{n-j} + (2/3)\sum_{j > n/2}\binom{n}{j}(.4)^j(.6)^{n-j}. \end{aligned}$$

For $n = 3$ this yields $P\{X \geq 2\} = .533$, and for $n = 5$ this yields $P\{X \geq 3\} = .526$. The student should request 3 examiners.

p. 181 #74. This problem involves Bernoulli trials, with parameter $p = \dfrac{12}{38}$ of success on one trial. One easily sees that

$$P\{\text{lose first 5 trials}\} = (1-p)^5 = .150\ ;$$

$$P\{\text{first win on fourth bet}\} = \left(\frac{26}{38}\right)^3\left(\frac{12}{38}\right) = .101. \quad (3\text{ losses, then a win})$$

Note also that the question concerns a geometric RV X of parameter $p = \frac{6}{19}$. (a) asks for $P(X > 5)$ and (b) for $P(X = 4)$, and these are given by the appropriate formulas for the geometric RV.

p. 181 #75. $P\{\text{stronger team wins in } i \text{ games}\} = P\{X = i\}$ where X, the number of games until the fourth win, is a negative binomial RV with parameters $p = .6$, $r = 4$. So for $i \geq 4$ we have $P\{X = i\} = \binom{i-1}{3}(.6)^4(.4)^{i-4}$. Numerical values:

i :	4	5	6	7
$P\{X = i\}$:	.1296	.2074	.2074	.1659

Thus the probability that the stronger team wins is the sum of these probabilities, 0.710208. Similarly, in a best-of-three series the probability that the stronger team wins is $(.6)^2 + 2(.6)^2(.4)) = .648$. Notice that we are dealing in this problem with the problem of the points with $r = m$, i.e., where the players need the same number of points (games) to win.

p. 181 #78. Let Y be the number of trials needed for 10 successes (heads) with a fair coin, so that Y is negative binomial with parameters $(10, 1/2)$. Since all but 10 of these Y trials are failures (tails), we have that the number of tails $X = Y - 10$. Thus

$$P\{X = k\} = P\{Y - 10 = k\} = P\{Y = k + 10\} = \binom{k+9}{9}\left(\frac{1}{2}\right)^{k+10}, \qquad k = 0, 1, 2, \ldots.$$

Chapter 4. Theoretical exercises: p. 183 #15(b). Let Y be the number of boys. If we know the total number n of children then Y is binomial, parameters $(n, .5)$, and $P\{Y = k\} = \binom{n}{k}2^{-n}$ if $n \geq k$. Thus we condition on the value of X (the number of children in the family):

$$\begin{aligned} P\{Y = k\} &= \sum_{n \geq k} P(\{Y = k\}|\{X = n\})P\{X = n\} \\ &= \sum_{n \geq k} \binom{n}{k} 2^{-n} P\{X = n\} \\ &= \sum_{n \geq k} \binom{n}{k} 2^{-n} \alpha p^n, \qquad \text{if } k \geq 1. \end{aligned}$$

It is possible to obtain a closed form for this summation using the formula (8.3) obtained by consideration of the negative binomial distribution. For simplicity (to avoid the special formula for $P\{X = 0\}$) we work out only the case $k \geq 1$. Then if we set $q = 1 - p/2$, so that $(1-q)/q = p/(2-p)$, and use the summation variable $m = n + 1$,

$$\begin{aligned} P\{Y = k\} &= \alpha \sum_{m \geq k+1} \binom{m-1}{(k+1)-1} \left(\frac{p}{2}\right)^{m-1} \\ &= \alpha \frac{2}{p} \sum_{m \geq k+1} \binom{m-1}{(k+1)-1} (1-q)^m \\ &= \frac{2\alpha}{p} \left(\frac{1-q}{q}\right)^{k+1} \sum_{m \geq k+1} \binom{m-1}{(k+1)-1} q^{k+1}(1-q)^{m-(k+1)} \\ &= \frac{2\alpha p^k}{(2-p)^{k+1}}, \end{aligned}$$

since the sum in the next to last equation has value 1, by (8.3).