

Midterm 2 Solutions
Math 250, Section B1
June 23, 2009

1. (6 pts.) Let $A = \begin{bmatrix} 4 & 2 & 2 & -3 \\ 3 & 0 & 0 & 0 \\ 6 & -1 & 4 & 1 \\ 2 & -1 & 0 & 3 \end{bmatrix}$.

(a) Compute the determinant of A .

Solution: Since the second row has only one nonzero entry, we do cofactor expansion along the second row:

$$\det A = (-3) \begin{vmatrix} 2 & 2 & -3 \\ -1 & 4 & 1 \\ -1 & 0 & 3 \end{vmatrix}.$$

To compute the 3×3 determinant, we do cofactor expansion along the third row, whence

$$\det A = -3 \left((-1) \begin{vmatrix} 2 & -3 \\ 4 & 1 \end{vmatrix} + (3) \begin{vmatrix} 2 & 2 \\ -1 & 4 \end{vmatrix} \right) = -3((-1)(14) + (3)(10)) = -48.$$

(b) Let $B = (\frac{1}{2}A)^{-1}$. Use your answer to part (a) to determine the determinant of B .

Solution: Since A is 4×4 , $\det(\frac{1}{2}A) = (\frac{1}{2})^4 \det A = -3$. Thus, $\det B = -\frac{1}{3}$.

2. (8 pts.) The matrix $A = \begin{bmatrix} 1 & 1 & 1 & -2 \\ 3 & 2 & 4 & 1 \\ 5 & 3 & 7 & 4 \\ -1 & 2 & -4 & 0 \end{bmatrix}$ has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
 Use this information to answer the following questions.

(a) Find a basis for the null space of A .

Solution: The general solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ is given by

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= x_3 \\ x_3 &\text{ free} \\ x_4 &= 0, \end{aligned}$$

so a basis for the null space of A is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(b) Find a basis for the row space of A .

Solution: A basis for the row space of A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (the

nonzero rows of the RREF of A).

(c) Find a basis for the column space of A^T .

Solution: Since the column space of A^T is the row space of A , a basis for the

column space of A^T is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(d) Is zero an eigenvalue of A ? Briefly explain your answer. (Hint: No computation required.)

Solution: Yes, since the nullity of A is one and therefore the system $A\mathbf{x} = \mathbf{0}$ has nonzero solutions.

3. (6 pts.) Let $V = \text{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 8 \\ 3 \end{bmatrix} \right\}$.

(a) Find a basis for V .

Solution: Let $A = \begin{bmatrix} 1 & -2 & 0 & -1 & -4 \\ -2 & 4 & 3 & 2 & 8 \\ -1 & 2 & 1 & 1 & 3 \end{bmatrix}$. The column space of A is equal

to V , so a basis for V is the set of pivot columns of A , which we find by reducing A to row echelon form:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & -4 \\ -2 & 4 & 3 & 2 & 8 \\ -1 & 2 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\substack{2r_1+r_2 \rightarrow r_2 \\ r_1+r_3 \rightarrow r_3}} \begin{bmatrix} 1 & -2 & 0 & -1 & -4 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{r_3 - \frac{1}{3}r_2 \rightarrow r_3} \begin{bmatrix} 1 & -2 & 0 & -1 & -4 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Thus the pivot columns of A are its first, third, and fifth columns, and a basis

for V is $\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 8 \\ 3 \end{bmatrix} \right\}$.

(b) Briefly explain why $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$ is NOT a basis for V .

Solution: We see from part (a) that $\dim V = 3$, so any basis for V must contain exactly 3 vectors.

4. (6 pts.) Suppose A is an $n \times n$ matrix. Correctly fill in each of the following blanks with exactly one of ROW SPACE, COLUMN SPACE, or NULL SPACE. You do NOT have to justify your answers.

- (a) A vector \mathbf{u} is in the NULL SPACE of A if and only if $A\mathbf{u} = \mathbf{0}$.
- (b) The scalar λ is an eigenvalue of A if and only if the NULL SPACE of $A - \lambda I_n$ contains nonzero vectors.
- (c) A vector \mathbf{u} is in the COLUMN SPACE of A if and only if the system $A\mathbf{x} = \mathbf{u}$ is consistent.
- (d) If $\det A = 0$, then the dimension of the NULL SPACE of A must be at least one.
- (e) The dimension of the row space of A plus the dimension of the NULL SPACE of A is equal to n .
- (f) Let R be the reduced row echelon form of A . The COLUMN SPACES of A and R are NOT necessarily equal.

5. (6 pts.) Suppose that A is a 6×6 matrix with no nonreal eigenvalues and exactly three real eigenvalues, 1, 2, and 3. Let W_1 , W_2 , and W_3 be the eigenspaces corresponding to 1, 2, and 3, respectively. List all possible characteristic polynomials of A that are consistent with the following information:

- (a) $\dim W_1 = 4$

Solution: The multiplicity of 1 must be at least four, and the multiplicities of 2 and 3 must each be at least one, so the only possibility is $(t-1)^4(t-2)(t-3)$.

- (b) $\dim W_1 = 3$

Solution: The multiplicity of 1 must be at least three, and the multiplicities of 2 and 3 must each be at least one, so there are three possibilities:

$$(t-1)^3(t-2)^2(t-3), \quad (t-1)^3(t-2)(t-3)^2, \quad \text{or} \quad (t-1)^4(t-2)(t-3)$$

- (c) $\dim W_1 = 2$ and $\dim W_2 = 2$

Solution: The multiplicities of 1 and 2 must each be at least two, and the multiplicity of 3 must be at least one, so there are three possibilities:

$$(t-1)^2(t-2)^2(t-3)^2, \quad (t-1)^3(t-2)^2(t-3), \quad \text{or} \quad (t-1)^2(t-2)^3(t-3)$$

6. (6 pts.) Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 9 & 3 & 10 \\ -5 & 0 & -2 \end{bmatrix}$.

- (a) Find the eigenvalues of A .

Solution: The characteristic polynomial of A is

$$\det(A - tI_3) = \begin{vmatrix} 3-t & 0 & 0 \\ 9 & 3-t & 10 \\ -5 & 0 & -2-t \end{vmatrix} = (3-t) \begin{vmatrix} 3-t & 10 \\ 0 & -2-t \end{vmatrix} = -(t-3)^2(t+2),$$

so the eigenvalues of A are 3 and -2 .

(b) Find a basis for each eigenspace.

Solution: The eigenspace corresponding to 3 is the null space of $A - 3I_3$.

Since the reduced row echelon form of $A - 3I_3$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (I'm skipping a bit of computation here, which you should fill in), a basis for the eigenspace corresponding to the eigenvalue 3 is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Since the reduced row echelon form of $A + 2I_3$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, a basis for the eigenspace corresponding to the eigenvalue -2 is $\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

7. (10 pts.) Determine whether each of the following statements is TRUE or FALSE. Justify each FALSE answer. (You don't have to justify TRUE answers.)

(a) If A is a 6×6 matrix and $\det A = 10$, then the columns of A are linearly independent.

Solution: TRUE (since $\det A \neq 0$, A is invertible and its columns are linearly independent)

(b) Every linearly independent subset of a subspace contains a basis for the subspace.

Solution: FALSE; e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a linearly independent subset of \mathcal{R}^2 that does not contain a basis for \mathcal{R}^2 . (Every linearly independent subset of a subspace is contained in a basis for the subspace.)

(c) If two matrices have the same characteristic polynomial, then they have the same eigenvalues.

Solution: TRUE (since the eigenvalues of a matrix are the roots of its characteristic polynomial)

(d) If A and B are both $n \times n$ matrices, then $\det(A + B) = \det A + \det B$.

Solution: FALSE. For example, if $A = I_2$ and $B = -I_2$, then $\det A$ and $\det B$ are each 1 but $\det(A + B) \neq 2$.

(e) $V = \text{Span} \left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ -3 \end{bmatrix} \right\}$ is a subspace of \mathcal{R}^3 of dimension two.

Solution: FALSE. V is a subspace of dimension *one*, since $V = \text{Span} \left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\}$.

8. (6 pts.) Note: The two parts of this problem are unrelated.

(a) Show that $W = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathcal{R}^3 : u_1 = u_2 u_3 \right\}$ is NOT a subspace of \mathcal{R}^3 .

Solution: Since $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are both in W but $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is not in W , we conclude that W is not closed under vector addition. (W is also not closed under scalar multiplication.)

(b) An $n \times n$ matrix B is called *idempotent* if $B^2 = B$. What can you say about the determinant of an idempotent matrix?

Solution: If B is idempotent, then

$$\det B = \det(B^2) = (\det B)^2,$$

so we must have $\det B = 0$ or $\det B = 1$. (Note that both are actual possibilities, since the $n \times n$ zero and identity matrices are idempotent.)