Midterm 1 Solutions
Math 250, Section B1
June 9, 2009

1. (8 pts.) The matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 0 & -4 & 1 \\ -1 & -3 & -7 & 0 \end{bmatrix}$ has reduced row echelon form $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Use this information to answer the following questions.

   (a) Determine the rank and nullity of A.
   Solution: The rank of A is 3, and the nullity of A is 1.

   (b) Is A invertible? Give a very brief explanation.
   Solution: No, since nullity$A > 0$. (A square matrix is invertible if and only if it has nullity equal to zero.)

   (c) Are the columns of A linearly independent? Give a very brief explanation.
   Solution: No, since nullity$A > 0$. (The columns of a matrix are linearly independent if and only if its nullity is zero.)

   (d) Does there exist a vector $\mathbf{b}$ in $\mathcal{R}^4$ such that the system $A\mathbf{x} = \mathbf{b}$ is inconsistent? Give a very brief explanation.
   Solution: Yes, since rank$A < 4$. ($A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ if and only if the rank of A is equal to its number of rows.)

2. (6 pts.) Determine whether the following system is consistent, and if so, find its general solution:

$$\begin{array}{rrrcr} x_1 & -2x_2 & +3x_3 & = & 9 \\ -2x_1 & +4x_2 & & = & -6 \\ -x_1 & +2x_2 & +x_3 & = & -1 \end{array}$$

Solution: We write down the augmented matrix for the system and use row operations to put it in reduced row echelon form:

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -2 & 4 & 0 & -6 \\ -1 & 2 & 1 & -1 \end{bmatrix} \xrightarrow[r_1+r_3 \to r_3]{2r_1+r_2 \to r_2} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 0 & 6 & 12 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{6}r_2 \to r_2} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 8 \end{bmatrix} \xrightarrow[r_3-4r_2 \to r_3]{r_1-3r_2 \to r_1} \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We conclude that the system is consistent, and the general solution is

$$\begin{aligned} x_1 &= 3 + 2x_2 \\ x_2 &\ \text{free} \\ x_3 &= 2 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

3. (6 pts.) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 5 \\ 1 & 3 & 1 \end{bmatrix}$. Compute A^{-1}. (Small hint: Remember that it's easy to check your answer.)

Solution: We start with $[A|I_3]$ and use row operations to put the left half in reduced row echelon form:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - r_1 \to r_3]{r_2 - 2r_1 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-r_3 \to r_3]{-r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \xrightarrow{r_1 - 3r_2 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -5 & 3 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[r_2 + r_3 \to r_2]{r_1 - 5r_3 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & 3 & 5 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

We conclude that $A^{-1} = \begin{bmatrix} -10 & 3 & 5 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$ (and we can check by verifying that $AA^{-1} = I_3$).

4. (6 pts.) Determine the values of r and s for which the system

$$\begin{array}{rrcl} 2x_1 & -x_2 & = & 3 \\ 4x_1 & +rx_2 & = & s \end{array}$$

has

(a) no solutions

Solution: We first form the augmented matrix and put it in row echelon form:

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & r & s \end{bmatrix} \xrightarrow{r_2 - 2r_1 \to r_2} \begin{bmatrix} 2 & -1 & 3 \\ 0 & r+2 & s-6 \end{bmatrix}$$

The system is inconsistent when $r + 2 = 0$ and $s - 6 \neq 0$, that is, when $r = -2$ and $s \neq 6$.

(b) exactly one solution

Solution: The system has exactly one solution when it's consistent and both variables are basic. This occurs when $r \neq -2$.

(c) infinitely many solutions

Solution: The system has infinitely many solutions when it's consistent and there are free variables. This occurs when $r = -2$ and $s = 6$.

5. (6 pts.) Determine whether the vector $\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$ is in the span of $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \right\}$.

If so, express $\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$ as a linear combination of vectors in $\mathcal{S}$. If not, briefly explain why.

Solution: The vector $\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$ is in the span of $\mathcal{S}$ if and only if the system

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$$

is consistent. We write the augmented matrix for this system and put it in row echelon form:

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & 1 & 3 & 2 \\ 0 & 2 & 4 & 6 \end{bmatrix} \xrightarrow{r_1+r_2 \to r_2} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 4 & 6 \end{bmatrix}$$

$$\xrightarrow{r_3-r_2 \to r_3} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

Thus, system is consistent, and its solution is $x_1 = 3, x_2 = -1, x_3 = 2$. We conclude that $\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$ is in the span of $\mathcal{S}$, and

$$\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}.$$

6. (16 pts.) Determine whether each of the following statements is `TRUE` or `FALSE`. Justify each `FALSE` answer by giving the corresponding correct statement or by providing a counterexample. (You don't have to justify `TRUE` answers.)

   (a) Every column of a matrix is a linear combination of its pivot columns.
   Solution: `TRUE`

   (b) If the reduced row echelon form of the augmented matrix of a consistent system of m linear equations in n variables contains k nonzero rows, then its general solution contains k free variables.
   Solution: `FALSE`. If the reduced row echelon form of the augmented matrix of a consistent system of m linear equations in n variables contains k nonzero rows, then its general solution contains k *basic* variables $n - k$ *free* variables.

   (c) A subset of $\mathcal{R}^n$ containing fewer than n vectors must be linearly independent.
   Solution: `FALSE`. For example, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a linearly dependent set of two vectors in $\mathcal{R}^3$.

   (d) If A and B are $m \times n$ matrices and B can be obtained from A by an elementary row operation, then there is an elementary $m \times m$ matrix E such that $B = EA$.
   Solution: `TRUE`

   (e) If A is an $m \times n$ matrix, then $A\mathbf{x} = \mathbf{0}$ is consistent if and only if $\operatorname{rank} A = n$.
   Solution: `FALSE`. The system $A\mathbf{x} = \mathbf{0}$ is *always* consistent, no matter what the of the rank of A (since $\mathbf{x} = \mathbf{0}$ is always a solution).

   (f) There are NO 3×5 matrices with rank 2 and nullity 3.
   Solution: `FALSE`. For example, the following 3×5 matrix has rank 2 and nullity 3:
   $$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (g) A set of two or more vectors in $\mathcal{R}^n$ is linearly dependent if and only if one of the vectors is a linear combination of the others.
   Solution: `TRUE`

   (h) If A and B are invertible $n \times n$ matrices, then $A + B$ is invertible.
   Solution: `FALSE`. For example, let $A = I_2$ and $B = -I_2$; then each of A, B is invertible, but $A + B$ is the 2×2 zero matrix and thus not invertible.

7. (5 pts.) Prove that if $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, then so is $\{\mathbf{u} + \mathbf{v}, 2\mathbf{v}\}$.
Solution: Let $\{\mathbf{u}, \mathbf{v}\}$ be linearly independent, and assume that
$$a(\mathbf{u} + \mathbf{v}) + b(2\mathbf{v}) = \mathbf{0}$$
for some scalars a and b. Then, rearranging, we have
$$a\mathbf{u} + (a + 2b)\mathbf{v} = \mathbf{0}.$$
Since $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, we must have $a = 0$ and $a + 2b = 0$, so $a = b = 0$. Therefore, $\{\mathbf{u} + \mathbf{v}, 2\mathbf{v}\}$ is linearly independent.