**Abstract**

This is an appendix from my dissertation, which in theory should be accessible to undergraduates with a background in Linear Algebra.

## 0.1 Algebras

**Definition 1.** *Given any two sets $A$ and $B$ the set $A \times B$ is the collection of pairs $(a, b)$ for $a \in A$ and $b \in B$.*

It is when we consider functions from $A \times B$ to a set $C$ that the first abuse of notation in this work occurs. We will write $f(a, b)$ instead of $f((a, b))$ in most instances.

**Definition 2.** *For vector spaces $V, W$ and $Z$, a map from $V \times W$ to $Z$ is bilinear if it satisfies the following rules:*
$f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$
$f(v, w_1 + w_2) = f(v, w_1) + f(v, w_2)$
$f(\lambda v, w) = f(v, \lambda w) = \lambda(v, w)$

**Definition 3.** *A not necessarily associative (or non-associative) algebra is a vector space $A$ equipped with a bilinear map, called product or multiplication, from $A \times A$ to $A$.*

We use juxtaposition to denote the image of elements under this product, writing $ab$ for the image of $(a, b)$ in the bilinear map of A.

**Definition 4.** *An associative algebra is a (not necessarily associative) algebra that has a distinguished vector (identity element) 1 with the property that $1a = a1 = a$ for $a \in A$ and if the associative law holds: $(ab)c = a(bc)$ for $a, b, c \in A$.*

*Example* 1. Consider the vector space given by basis elements $\{1(= x^0), x, x^2, x^3, \ldots\}$. We can create an algebra structure by defining the product of $x^i$ and $x^j$ to be $x^{i+j}$ and extending linearly. This is the well known algebra of polynomials in the variable x.

*Example* 2. Consider the vector space $M_n$ of $n$ by $n$ matrices over a field $\mathbb{F}$. This is an associative algebra under the product of matrix multiplication.

*Example* 3. The cross product is a well known non-associative algebra on the vector space $\mathbb{R}^3$.

Unless otherwise specified, the term *algebra* will be used to mean associative algebra throughout this work. Notice that with the terminology we have set up, all associative algebras are non-associative algebras, though the converse does not hold.

**Definition 5.** *For (not necessarily associative) algebras $A$ and $B$, a linear map $f : A \to B$ is a homomorphism if $f(ab) = f(a)f(b)$ for $a, b \in A$ and $f(1) = 1$. A homomorphism is an isomorphism if the map is also a bijection. Two algebras $A$ and $B$ are isomorphic if there exists an isomorphism between them.*

## 0.2 Tensor Products and Direct Sums

Our next goal is to define the free associative algebra $T(V)$ of a vector space $V$, but before we can begin we need to explain the following constructions on vector spaces.

**Definition 6.** *The direct sum of two vector spaces $V$ and $W$ is a vector space $V \oplus W$ equipped with a map $i : V \oplus W \to V \times W$ which is universal: That is for any vector space $U$ and map $f : U \to V \times W$ there exists a unique map $u$ from $U$ to $V \oplus W$ so that $i \circ u = f$.*

One can then set up a dual construction as follows:

**Definition 7.** *The tensor product of two vector spaces $V$ and $W$ is a vector space $V \otimes W$ equipped with a bilinear map $b : V \times W \to V \otimes W$ which is universal: for any bilinear map $\beta : V \times W \to U$ to a vector space $U$, there is a unique linear map $u$ from $V \otimes W$ to $U$ so that $u \circ b = \beta$.*

Uniqueness of these constructions follows from the universal definitions. For existence we turn to the less natural but more direct constructions of these vector spaces: If $\{e_i\}$ and $\{f_j\}$ are bases for $V$ and $W$, then the elements $\{e_i\} \cup \{f_j\}$ form a basis for $V \oplus W$ and the elements $\{e_i \otimes f_j\}$ form a basis for $V \otimes W$.

These constructions are commutative:
$V \oplus W \cong W \oplus V$
$V \otimes W \cong W \otimes V$
associative:
$(U \oplus V) \oplus W \cong U \oplus (V \oplus W)$
$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$
and distributive:
$(U \oplus V) \otimes W \cong (U \otimes W) \oplus (V \otimes W)$.

Because of the associative nature of the tensor product we can consider the *tensor powers* $V^n = V^{\otimes n} = \underbrace{V \otimes V \otimes \ldots \otimes V}_{ntimes}$ of a fixed vector space for any $n \in \mathbb{N}$. Here $V^0$ is our base field $\mathbb{F}$ and $V^1 = V$.

We can similarly consider the direct sums $V^{\oplus n} = \underbrace{V \oplus V \oplus \ldots \oplus V}_{ntimes}$ of a fixed vector space $V$ for any $n \in \mathbb{N}$ if we treat each of the separate $V$'s as being disjoint. We also can define the direct sum of countably infinitely many vector spaces $\{V_0, V_1, V_2, \ldots\}$. The space $\bigoplus V_i = \bigoplus_0^\infty V_i$ is the set of elements in $\bigoplus_0^n V_i$ for some $n \in \mathbb{N}$. The addition of $x \in \bigoplus_0^n V_i$ and $y \in \bigoplus_0^m V_i$ simply takes place in $\bigoplus_0^k V_i$ where $k = max\{m, n\}$.

*Example* 4. Given a 1-dimensional vector space $V = span\{v\}$, the vector space $V^{\otimes n}$ is a 1-dimensional vector space spanned by $v \otimes v \otimes \ldots \otimes v$. The vector space $V^{\oplus n}$, however, has dimension $n$.

## 0.3 Free Algebras

We are now ready to discuss the following:

**Definition 8.** *The free associative algebra of a vector space $V$ is an associative algebra $T(V)$ together with an injective map $i : V \to T(V)$ so that for any associative algebra $A$ and linear map $j : V \to A$ there is a unique algebra map $f : T(V) \to A$ so that $f \circ i = j$.*

Uniqueness again follows from the universal definition. For existence, we have the following construction. $T(V)$ is the algebra on the vector space $T(V) = V^0 \oplus V^1 \oplus V^2 \oplus \ldots = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots$ with a product given as follows. Set $(v_1 \otimes v_2 \otimes \ldots \otimes v_m)(w_1 \otimes w_2 \otimes \ldots \otimes w_n) = v_1 \otimes v_2 \otimes \ldots \otimes v_m \otimes w_1 \otimes w_2 \otimes \ldots \otimes w_n$ in $V^{n+m}$ for $v_i, w_j \in V$ and extend linearly to all of $T(V)$.

*Example* 5. Let $V$ be the 1-dimensional vector space spanned by a vector $x$. Then $T(V)$ is isomorphic to the familiar polynomial algebra in one variable $\mathbb{F}[x]$ under the map that sends $x^{\otimes n}$ to $x^n$.

*Example* 6. Let $V$ now be the 2-dimensional vector space spanned by the vectors $x$ and $y$. Here, however, $T(V)$ is not isomorphic to the algebra of polynomials in two variables $\mathbb{F}[x, y]$. The obvious candidate for such a map (sending an element of $V^{\otimes k}$ with $n$ $x$'s and $k-n$ $y$'s to $x^n y^{k-n}$) produces a homomorphism but not an isomorphism because in $V \otimes V, x \otimes y$ is not equal to $y \otimes x$ (yet both get mapped to the polynomial $xy$).

## 0.4 Presented Algebras

Most of the algebras we look at throughout this work are expressed as quotients of a free associative algebra on some vector space $V$.

**Definition 9.** *A subspace $I$ of an algebra $A$ is an ideal if for any $a \in A, x \in I$ both $xa$ and $ax$ are in $I$.*

*Example* 7. Let $\mathbb{F}[x]$ be the polynomial algebra in variable $x$. For any $k \in \mathbb{N}$ the subspace $V_k$ spanned by $x^j$ for $j \geq k$ is an ideal in $\mathbb{F}[x]$.

**Definition 10.** *Let $A$ be an algebra and $X = \{x_i\}$ a collection of elements of $A$. The ideal generated by $X$ is the smallest ideal in $A$ containing $X$ (the intersection of all ideals in $A$ containing $X$).*

*Example* 8. Let $v$ be any element of the vector space $T(V)$. The ideal generated by $v$ is simply the subspace spanned by all vectors in $T(V)$ of the form $avb$ for $a, b \in T(V)$. An arbitrary element of this ideal will be of the form $\sum_i a_i v b_i$.

**Definition 11.** *Given a vector space $V$ with subspace $W$, the vector space $V/W$ is the vector space where the elements are subspaces of the form $(v+W)$ for $v \in V$ with operations $(v+W)+(x+W) = (v+x+W)$ and $\lambda(v+W) = (\lambda v + W)$.*

**Definition 12.** *Given an algebra $A$ with an ideal $I$ we can form the quotient algebra $A/I$ as follows: The elements of $A/I$ are the subspaces in $A$ of the form $a+I$ for some $a \in A$. The product of $a+I$ and $b+I$ is the element $ab+I$ in $A/I$. If $A$ is an associative algebra the $A/I$ will be associative as well with identity element $1+I$ where $1$ is the unit in $A$.*

**Definition 13.** *Given a set $X = \{x_i\}$ let $V$ be the vector space with basis elements $X$. Suppose $R = \{r_j\}$ is a collection of elements of $T(V)$ and $I$ is ideal in $T(V)$ generated by $R$. Then the algebra $A$ presented by generators $X$ and relations $R$ is the algebra $T(V)/I$.*

*Note.* We will usually write the relations for a presented algebra in product form. For example $xy - yx$ instead of $x \otimes y - y \otimes x$. Since the two are the same, this does not create any difficulty and saves a small amount of space. Perhaps a more serious abuse of notation is that we will often refer to the element $x + I$ of $T(V)/I$ as $x$.

*Example* 9. Consider the algebra $A$ given by generators $x$ and $y$ with single relation $xy - yx$ (which sits inside $V^2 = V \otimes V$ in $T(V)$). The map $f$ from $\mathbb{F}[x, y]$ to $A$ we get by sending $x^k y^j$ to $\underbrace{x \otimes x \ldots \otimes x}_{ktimes} \otimes \underbrace{y \otimes \ldots \otimes y}_{jtimes}$ and extending linearly is an isomorphism from $\mathbb{F}[x, y]$ to $A$.

*Example* 10. Let $V$ be a vector space with basis $\{x_i\}_{i \in I}$ for some indexing set $I$. The algebra generated by $\{x_i\}$ with relations $x_i x_j - x_j x_i$ for $i, j \in I$ is sometimes called the symmetric algebra on $V$ or $S(V)$. This algebra has a universal property as well: There is an injective map $i : V \to S(V)$ so that given any associative commutative algebra $A$ and linear map $j : V \to A$, there is a unique map $f : S(V) \to A$ so that $f \circ i = j$.

$S(V)$ is isomorphic to the algebra of polynomials in variables $\{x_i\}_{i \in I}$.

*Example* 11. Let $V$ be a vector space with basis $\{x_i\}_{i\in I}$ for some indexing set $I$. The algebra generated by $x \otimes x$ for all $x \in V$ is sometimes called the exterior algebra on $V$ or $E(V)$. Once again this algebra can be described by a universal property: There is an injective map $i : v \to E(V)$ so that given any linear map $j : V \to A$ into an algebra $A$ so that $j(x)^2 = 0$, there is a unique map $f : E(V) \to A$ so that $f \circ i = j$.

**Definition 14.** *A nonzero relation $r$ in $T(V)$ is said to have degree $k$ if it sits inside $V^{\otimes k}$ in $T(V)$ for some $k \in \mathbb{N}$.*

*Example* 12. Consider an algebra $A$ with generator $a$ and a single relation sitting inside the space $V \otimes V (\in T(V))$. This relation must be of the form $\lambda x \otimes x$ and hence $x^2$ is zero in $A$. To see this remember that when we write $x^2$ we mean "$x$ times $x$" which in $A$ is really $(x + I)(x + I) = x^2 + I = 0 + I$ because $x^2$ is in $I$ (since $I$ is a subspace, $x^2$ is in $I$ if and only if $\lambda x^2$ is in $I$). Since $I$ is an ideal, we know $x^j$ times $x^2$ must be in $I$ for all $j \in \mathbb{N}$ and hence $x^k = 0$ for any $k \geq 2$. We see that $A$ is a vector space of dimension 2, a linearly independent set of $A$ is given by $1 + I, x + I$, and we have the following multiplication table describing the structure here:

| $\cdot$ | 1 | $x$ |
|---|---|---|
| 1 | 1 | $x$ |
| $x$ | $x$ | 0 |

*Example* 13. Consider an algebra A with generators $a$ and $b$ and a single relation of the form $\lambda a + \mu b$ (for $\lambda, \mu \in \mathbb{F}$ not both equal to zero). Assume, without loss of generalization, that $\mu$ is nonzero. Then the map $f$ from $\mathbb{F}[x]$ to $A$ sending $x^n$ to $b^n$ is an isomorphism. Hence any algebra of this form is isomorphic to $\mathbb{F}[x]$

**Definition 15.** *A quadratic algebra is an algebra presented by generators and relations where all relations are of degree two.*

## 0.5 Graded Algebras

Some of the algebras we have considered in this appendix have been finite dimensional, but others like $T(V)$ and $\mathbb{F}[x]$ are infinite dimensional. In order to better understand the structure of infinite dimensional algebras, it will help to have some way of discussing their "size" other than just looking at the dimension of the entire vector space. With that in mind we begin the next set of definitions.

**Definition 16.** *Let $S$ be a set. A vector space $V$ is said to be $S$-graded if it is the direct sum $V = \bigoplus_{\alpha\in S} V_\alpha$ of disjoint subspaces $V_\alpha$. The elements in $V_\alpha$ are called homogeneous of degree $\alpha$ and $V_\alpha$ is called the homogeneous subspace of degree $\alpha$. For $v \in V_\alpha$ we write deg $v = \alpha$. Given two $S$-graded vector spaces $V$ and $W$ a linear map $f : V \to W$ is grading-preserving if $f(V_\alpha) \subseteq W_\alpha$ for all $\alpha \in S$.*

*Example* 14. Pick $n \in N$ and let $V$ be the $n$-dimensional vector space with basis $\{b_1, b_2, \ldots, b_n\}$. One way to grade this space is to set $S = [n] = \{1, 2, \ldots, n\}$ and let $V_i = span\{b_i\}$. Then each vector of the form $\lambda b_i$ is homogeneous of degree $i$, yet no other vector are homogeneous. The original vector space has dimension $n$ but we have split it up into 1-dimensional subspaces. Our doing this may seem slightly arbitrary, as there are many different ways we could have split $V$ up (for example letting $V_i = span\{b_1 + b_2 + \ldots + b_i\}$ would have produced another grading). When our vector space has some other structure, then we will expect our gradings to behave well with respect to that structure.

Next we would like to extend an $S$-grading of $V$ to an $S$-grading of the space $V/W$ where $W$ is a subspace of $V$. The natural idea would be to define $(V/W)_i = \{(v+W) \in V/W | v \in V_i\}$. This does not always work though as is shown in the next example.

*Example* 15. Consider the vector space $V = span\{b_1, b_2\}$ with grading $V_1 = span\{b_1\}, V_2 = span\{b_2\}$. Let $W$ be the subspace $W = span\{w\}$ where $w = b_1 + b_2$. We can attempt to grade W by setting $(V/W)_i = \{(v+W) \in V/W | v \in V_i\}$, but then $(V/W)_1 = \{(v+W) \in V/W | v \in V_1\} = \{\lambda b_1 + W | \lambda \in \mathbb{F}\}$ which is all of $V/W$ (since every vector in $V$ is in the span of $b_1$ and $w$). Similarly, $(V/W)_2$ is also equal to all of $V/W$ so we haven't graded $V/W$ as our subsets $(V/W)_i$ are not disjoint.

We will need the next definition to make gradings of quotients work.

**Definition 17.** *Let $V$ be an $S$-graded vector space. A subspace $W$ is a graded subspace if $W = \bigoplus_{\alpha \in S} W_\alpha$ where $W_\alpha = W \cap V_\alpha$.*

*Example* 16. Consider the set $S = \{1, 2\}$ and S-graded vector space $V = span\{b_1, b_2\}$ with grading $V_i = span\{b_i\}$. The subspace $W = span\{b_1\}$ is graded with $W_1 = W, W_2 = 0$. The subspace $W = span\{b_1 + b_2\}$ is not graded as $W_1 = W \cup V_1$ and $W_2 = W \cup V_2$ are both equal to zero (and hence $W \neq W_1 \oplus W_2$).

**Proposition 0.1.** *Suppose $V$ is an $S$-graded vector space and $W$ is a subspace of $V$. Set $(V/W)_i = \{(v+W) | v \in V_i\}$. Then $V/W$ is an $S$-graded vector space with homogeneous spaces $(V/W)_i, i \in S$ if and only if $W$ is a graded subspace of $V$.*

*Proof.* We prove this for the case where our index set $S$ is finite for notational purposes though the infinite case is similar. By renaming, we can assume $S = [n]$ for some $n \in \mathbb{N}$.

First consider the statement: $W$ is a graded subspace of $V$. This means that $W$ is the direct sum of the spaces $W \cap V_i$ which is equivalent to saying any $w \in W$ is in that direct sum. We know (since $V$ is $S$-graded) that we can express any element $w$ in $W$ as $w = v_1 + v_2 + \ldots + v_k$. The vector $w = v_1 + v_2 + \ldots + v_k$ can only be in the direct sum $\bigoplus_i W \cap V_i$ if $v_i \in W$ for each $i$.

So $W$ is a graded subspace of $V$ if and only if ($\heartsuit$) for every $v_1 + v_2 + \ldots + v_k \in W$ with $v_i \in V_i$ we have $v_i \in W$.

Now consider the statement: $V/W$ is an $S$-graded vector space with homogeneous spaces $(V/W)_i, i \in S$. It is always true that the sets $V_i + W$ span $V/W$ (since $V/W$ is the set of all $(v+W)$ and the $V_i$ span $V$). So this statement is equivalent to saying that every vector in $V/W$ is expressible as a sum of the $(V/W)_i$'s in only one way. That means if $v = (x_1 + W) + (x_2 + W) + \ldots + (x_n + W) = (y_1 + W) + (y_2 + W) + \ldots + (y_n + W)$ for $x_i + W, y_i + W \in (V/W)_i$ then $(x_i + W) = (y_i + W)$ for each $i \in [n]$. That is equivalent to the statement: if $x_1 + \ldots + x_n + W = y_1 + \ldots + y_k + W$ for $x_i, y_i \in V_i$ then $(x_i + W) = (y_i + W)$ for each $i \in [n]$. We see that $x_1 + \ldots + x_n + W = y_1 + \ldots + y_k + W$ exactly when $(x_1 - y_1) + \ldots + (x_n - y_n) \in W$. We can set $z_i = x_i - y_i$ to give us the following statement: If $z_1 + z_2 + \ldots + z_n \in W$ for $z_i \in V_i$ then $z_i \in W$ for all $i \in [n]$.

Since this statement is equivalent to the statement $\heartsuit$ above we are done.

□

Up until now we have been allowed to assume $S$ is just any arbitrary set. When we allow $S$ to be an abelian group we can create some interesting graded structures.

In the case where $S$ is an abelian group and $V$ and $W$ are $S$-graded vector spaces, we can give $V \otimes W$ a unique $S$-grading by setting $(V \otimes W)_\gamma = \sum_{\alpha+\beta=\gamma} V_\alpha \otimes V_\beta$ for each $\gamma \in S$. We can extend this reasoning to grade an arbitrary number of tensor factors by setting $(V^{\otimes n})_\gamma = \sum_{\alpha_1+\alpha_2+\ldots+\alpha_n=\gamma} V_{\alpha_1} \otimes V_{\alpha_2} \otimes \ldots \otimes V_{\alpha_n}$.

*Example* 17. Consider the $\mathbb{N}$ graded vector space $V$ where $V_1 = V$ and $V_i = 0$ for $i \neq 1$ (so all elements of $V$ are homogeneous of degree 1). Then the space $V \otimes V$ inherits a grading where all elements are homogeneous of degree two, and the space $V^{\otimes n}$ inherits a grading where all elements are homogeneous of degree $n$.

*Example* 18. Now consider a two dimensional vector space $V$ with basis $\{b_i, b_2\}$. Consider the $\mathbb{N}$-grading where $V_1 = span\{b_1\}, V_2 = span\{b_2\}, V_i = 0$ for $i \notin \{1, 2\}$. Then in $V \otimes V$ we get a grading where $(V \otimes V)_2 = span\{b_1 \otimes b_1\}$, $(V \otimes V)_3 = span\{b_1 \otimes b_2, b_2 \otimes b_1\}$, $(V \otimes V)_4 = span\{b_2 \otimes b_2\}$, and $(V \otimes V)_i = 0$ otherwise.

**Definition 18.** *Now let $S$ be an abelian group and $A$ be a (not necessarily associative) algebra. Then $A$ is an $S$-graded algebra if $A$ is $S$-graded as a vector space and if $A_\alpha A_\beta \subset A_{\alpha+\beta}$ for all $\alpha, \beta \in S$.*

*Example* 19. Now we get back to work on a most important example, the free associative algebra $T(V) = \mathbb{F}1 \oplus V \oplus (V \otimes V) \oplus \ldots$. The $\mathbb{N}$-grading we consider for this algebra arises from the condition that deg $V = 1$. Then the product of any two nonzero elements in $V$ must have degree two hence the elements of $V \otimes V$ are all homogeneous of degree two. By continuing with products we see that $V^{\otimes n}$ must have degree $n$. Note that this is the same as the degree we would assign considering $V^{\otimes n}$ as graded tensor product of vector spaces where deg $V = 1$. This space $\mathbb{F}1$ can only be degree 0, since $\mathbb{F}1$ times $V$ maps to $V$.

This standard $T(V)_n = V^{\otimes n}$ grading is important because any we can grade any presented algebra with a homogeneous space of relations by assigning it a grading as quotient vector space.

**Proposition 0.2.** *Let $A$ be an algebra presented by generators $G$ and relations $R$ where each relation is $R$ is homogeneous of some degree. Then the ideal $I$ generated by $R$ is a homogeneous subspace of $T(V)$ and the grading we get from treating $T(V)/I$ as a graded quotient vector space gives $A$ the structure of a graded algebra.*