The Algebra P_n

David Nacin

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Abstract

We define algebras which correspond to graphs on n nodes. We then examine the algebra corresponding to the graph P_n, the n vertex path. We show it is a Koszul algebra, find a basis, and discuss its Hilbert series.

1 Introduction and Construction

We begin by defining a class of algebras parameterized by the natural numbers. Let $\{1, 2, 3, \ldots, n\}$ be denoted by $[n]$.

The Algebras Q_n defined in [3] are given by generators $u(A), \emptyset \neq A \subset [n]$ and relations

$$\sum_{C,D \subset A} [u(C \cup i), u(D \cup j)] = (\sum_{E \subset A} u(E \cup i \cup j)) \sum_{F \subset A} (u(F \cup i) - u(F \cup j))$$

for all $A \subset [n], i, j \in [n] \setminus A, i \neq j$.

Definition 1. *A complex with n nodes is a family $\mathcal{F}$ of nonempty subsets $A \subset [n]$ satisfying $A \in \mathcal{F}, B \subset A \Rightarrow B \in \mathcal{F}$. The dimension of $\mathcal{F}$ is defined as $dim\mathcal{F} = max_{A \in \mathcal{F}}(|A| - 1)$*

Definition 2. *Let $\mathcal{F}$ be a complex with n nodes. Define $Q_n(\mathcal{F})$ to be the quotient algebra of Q_n by the ideal generated by the elements u(A) for all $A \notin \mathcal{F}$.*

Notice that for any complex $\mathcal{F}$, $Q_n(\mathcal{F})$ has a presentation as a quadratic algebra.

Example. If $\mathcal{F} = P([n]) - \emptyset$ then $Q_n(\mathcal{F}) \cong Q_n$.

Example. If $\mathcal{F} = \{A \subset \mathcal{F} || A| = 1\}$ then $Q_n(\mathcal{F})$ is isomorphic to the algebra of commutative polynomials in n variables.

If $\mathcal{F}' \subset \mathcal{F}$ is a subcomplex then $Q_n(\mathcal{F}')$ is naturally isomorphic to a quotient algebra of $Q_n(\mathcal{F})$.

Let $n_1 < n_2$ and let $\mathcal{F}$ be a complex with n_1 nodes. Then as $[n_1] \subset [n_2]$, $\mathcal{F}$ may be viewed as a complex with n_2 nodes. We denote this complex by $\mathcal{F}'$. Then $Q_{n_1}(\mathcal{F}) \cong Q_{n_2}(\mathcal{F}')$ since every generator u(A) of Q_{n_2} with $A \not\subseteq [n_1]$ is outside $\mathcal{F}'$. Consequently every algebra $Q_n(\mathcal{F})$ occurs, up to isomorphism, for a complex $\mathcal{F}$ containing every i, $1 \leq i \leq n$.

Consider the case where $\mathcal{F}$ is a complex of dimension one with n nodes. We can then also look at $\mathcal{F}$ as a graph on n nodes. To do this define V, our set of vertices, to be the set of elements of $\mathcal{F}$ with cardinality one. Our set of edges, E is the set of elements of $\mathcal{F}$ with cardinality two. We adopt the convention of considering a graph to have no loops or multiple edges. Then there is actually a one to one correspondence between graphs on n vertices and complexes with n nodes and dimension one.

Theorem 1. *[2] Let $\mathcal{F}$ be a complex with n nodes and dimension one. Then the algebra $Q_n(\mathcal{F})$ is generated by the elements $u(i)$ for $i \in [n]$ and $u(i,j)$ for $\{i,j\} \in E$ with the following relations (assume $u(i,j) = 0$ if $\{i,j\} \notin E$):*
$(i)[u(i), u(j)] = u(i,j)(u(i) - u(j))$ $i \neq j,\ i,j \in [n]$
$(ii)[u(i,k), u(j,k)] + [u(i,k), u(j)] + [u(i), u(j,k)] = u(i,j)(u(i,k) - u(j,k))$ for distinct $i,j,k \in [n]$
$(iii)[u(i,j), u(k,l)] = 0$ for distinct $i,j,k,l \in [n]$

If $\mathcal{F}$ is the complex of dimension 1 corresponding to a graph we will write $Q_n(G) = Q_n(\mathcal{F})$. We refer to the elements u(i) $i \in [n]$ as nodes and $u(i,j)$ $\{i,j\} \in E$ as edges. We also often denote $u(i,j)$ as $u(ij)$. Using this terminology the following proposition is immediate from (i) and (iii).

Proposition 1.1. *Nodes in G commute if they are not connected by an edge. Non-adjacent lines in G commute.*

It is harder to find a way to simplify relation (ii). To gain some insight first fix distinct i,j,k in $[n]$. Let $V = \text{span}\{u(i), u(j), u(k), u(i,j), u(j,k), u(i,k)\}$ and let $v_{i,j,k} = [u(i,k), u(j,k)] + [u(i,k), u(j)] + [u(i), u(j,k)] - u(i,j)(u(i,k) - u(j,k))$. Consider the natural action of S_3 (the permutation group on three letters) on $T(V)$ defined by setting $\sigma u(i) = u(\sigma(i))$ and $\sigma u(i,j) = u(\sigma(i), \sigma(j))$ and extending linearly.

Proposition 1.2. *The orbit of $v_{i,j,k}$ under the action of S_3 spans a space of dimension two in $T(V)$.*

Proof. Let μ be the permutation given by $i \to j \to i$ and τ be given by $i \to k \to i$. Since these permutations generate S_3 it will be enough to show the action of τ and μ sends the space $\text{span}\{v_{i,j,k}, v_{k,j,i}\}$ back to itself. Since this space is clearly fixed by τ we need only worry about μ. A short computation shows that μ sends $v_{i,j,k}$ to $-v_{i,j,k}$ and $v_{k,j,i}$ to $v_{k,j,i} - v_{i,j,k}$. □

We say G is triangle free if $\{i,j\}, \{j,k\} \in E \implies \{i,k\} \notin E$ for any $\{i,j\} \neq \{j,k\}$. In this case relation (ii) simplifies further.

Proposition 1.3. *If G is a triangle free graph then (ii) is equivalent to*
(ii') $u(i)$ commutes with $u(j,k)$ whenever $\{j,k\} \in E, \{i,j\}, \{i,k\} \notin E$ and
(ii'') $[u(i),u(jk)]+u(ij)u(jk) = [u(k), u(i,j)] + u(j,k)\, u(i,j) = 0$ whenever $\{i,j\}, \{j,k\} \in E, \{i,k\} \notin E$
for any distinct $i,j,k \in [n]$.

Proof. We know from our last proposition that we can replace relation (ii) with $v_{i,j,k}$ and $v_{k,j,i}$. In the situation where $\{i,j\}, \{i,k\} \notin E$, $v_{i,j,k}$ becomes $[u(i), u(j,k)]$ and $v_{k,j,i}$ becomes zero since $u(i,j)$ and $u(i,k)$ are zero. This gives us the relation (ii').

In situation (ii'') $u(i,k) = 0$ so $v_{i,j,k} = [u(i), u(jk)] + u(ij)u(jk)$ and $v_{k,j,i} = [u(k), u(ij)] + u(jk)u(ij)$ so we are done. □

Now let us specialize to one particular case of triangle free graph, the n vertex path P_n given in hypergraph notation by the complex

$$\{\{1\}, \{2\}, \ldots, \{n\}, \{1,2\}, \{2,3\}, \ldots, \{n-1, n\}\}.$$

To make our notation simpler we will identify vertices and edges of the graph with the corresponding generators for the algebra. Thus we refer to the elements of our algebra P_n(which is really $Q_n(P_n)$) by v_i for $u(i)$ and e_{ij} for $u(i,j)$. Applying everything we have shown about the relations (i), (ii) and (iii) we get the following proposition.

Proposition 1.4. *The algebra P_n generated by the complex(graph) P_n is presented by generators $v_1, v_2, \ldots, v_n, e_{12}, e_{23}, \ldots, e_{n-1,n}$ and relations*

$[v_i, v_j] = 0$ *for* $j > i+1,\ i,j \in [n]$
$[v_i, v_{i+1}] + e_{i,i+1}(v_{i+1} - v_i) = 0$ *for* $i \in [n-1]$
$[e_{i,i+1}, e_{j,j+1}] = 0$ *for* $j > i+1,\ i,j \in [n-1]$
$[v_i, e_{j,j+1}] = 0$ *if* $j > i+1$ *or* $j < i-2,\ j \in [n-1], i \in [n]$
$[v_i, e_{i+1,i+2}] + e_{i,i+1}e_{i+1,i+2} = 0,\ i \in [n-2]$
$[v_{i+2}, e_{i,i+1}] + e_{i+1,i+2}e_{i,i+1} = 0,\ i \in [n-2]$

2 ch(P_n) and pc-algebras

Let V be the span of the generators of P_n. We define an increasing filtration of $T(V)$ as follows. Set $G^{(0)} = \mathbf{F}1$ and $G^{(i)} = \text{span}\,\{u(A_1)u(A_2)\ldots u(A_k)|\sum_{l=1}^{k} 3 - |A_l| < i\}$ for $i \geq 1$. This induces a filtration of our algebra P_n and hence we can consider the associated graded algebra gr(P_n). If we chop off the non-commutator terms in the relations given in the last proposition, those relations will all hold true in gr(P_n). However, there is no reason at this time to think that these are enough to present gr(P_n). This does not stop us from considering the algebra given by these chopped relations. We call it ch(P_n).

Definition 3. *The algebra ch(P_n) is presented by generators $v_1, v_2, \ldots, v_n, e_{12}, e_{23}, \ldots, e_{n-1,n}$ and relations*

$[v_i, v_j] = 0$ *for* $j > i+1,\ i,j \in [n]$
$[v_i, v_{i+1}] = 0$ *for* $i \in [n-1]$
$[e_{i,i+1}, e_{j,j+1}] = 0$ *for* $j > i+1,\ i,j \in [n-1]$
$[v_i, e_{j,j+1}] = 0$ *if* $j > i+1$ *or* $j < i-2,\ j \in [n-1], i \in [n]$
$[v_i, e_{i+1,i+2}] = 0,\ i \in [n-2]$
$[v_{i+2}, e_{i,i+1}] = 0,\ i \in [n-2]$

Our intention is to use Bergman's Diamond Lemma [1] to find a basis for ch(P_n) and later gr(P_n). The idea is to pick a set of generators, come up with an ordering on the set of monomials in our algebra, and turn the set of relations for our algebra into a set of reductions. We can then use these reductions to rewrite any monomial in an element of our algebra. For example, if our ordering says the monomial v_3v_1 is higher than v_1v_3 then we can rewrite the relation $[v_1, v_3] = 0$ as $v_3v_1 = v_1v_3$ and replace any occurrence of v_3v_1 with v_1v_3. It is easy to see that a basis with no monomial containing the string v_3v_1 must exist. We can then write up a list of "bad" words and try and find a basis amongst the monomials containing none of these words.

Problems might occur when one word can be reduced in more than one way. Sometimes two different reductions can be applied to the same monomial to give us two different elements of our algebra. We call these situations ambiguities. If we can continue through our set of reductions to reduce both elements down to the same thing we say the ambiguities resolves. The diamond lemma says that if all ambiguities resolve then we have found a basis for our algebra in the set of monomials containing no bad words.

Right now we can only see that gr(P_n) is a quotient of ch(P_n). We will soon show that the two are indeed equal. In order to show equality in such a situation it is enough to show that these two algebras have the same Hilbert series. Before applying the diamond lemma to ch(P_n) we shall first prove some more general results about algebras whose relations are all commutators of generators. Then we can see how these results will apply here. We start with the following definition.

Definition 4. *Suppose an algebra A has a presentation by generators $\{a_1, a_2, \ldots, a_n\}$ and some relation set R where each relation in R is of the form $[a_i, a_j] = 0$ for some $i, j \in [n]$. We then call A a pre-generator-commuting algebra (or a PGC-algebra for short).*

These algebras, also called free partially commuting algebras, have been studied before in [DK].

Notice that if A is a PGC-algebra on n generators then for some ideal I in A, $A/I = S_n$ the free commutative algebra on n generators. The structure of a PGC-algebra is based entirely on which pairs of generators commute. We can then describe such an algebra using a graph with a node to represent each generator and edges between two nodes if the generators commute.

Definition 5. *Let A be a PGC-algebra on generators $\{a_1, a_2, \ldots, a_n\}$. The commuting graph $G_c(A)$ of A is the graph with nodes labeled $\{a_1, a_2, \ldots, a_n\}$ and edges given by the rule e_{a_i,a_j} is an edge if and only if $[a_i, a_j] = 0$. We often also consider the complement of this graph, $\overline{G_c(A)}$. We call this the non-commuting graph of A since an edge exists between two generators only when they do not commute.*

Example. A simple example of a PGC-algebra is the algebra presented by generators $\{a, b, c\}$ and relations $ab - ba = bc - cb = 0$. The graphs $G_c(A)$ and $\overline{G_c(A)}$ are shown here:

The diamond lemma gives a method for determining a basis for such an algebra. If, as in the example above, we choose a monomial ordering given first by length and then lexicographically with $c > b > a$ we get the following reductions:

$cb = bc$

$ba = ab$

This gives us one ambiguity, namely cba, that we must resolve.

$c(ba) = cab$ and $(cb)a = bac$ which gives us the new reduction $cab = bac$ and one new ambiguity to resolve, caba.

$ca(ba) = caab$ and $(cab)a = baca$ which gives us the new reduction $caab = baca$ and the new ambiguity caaba.

We can inductively show that by adjoining the reductions $caa\ldots ab = baca\ldots a$ we can resolve all ambiguities and we end up with the following complicated list of bad words: $cb, ba, cab, caab, caa\ldots ab$. A basis for the algebra consists of the set of all monomials not containing one of these strings.

Example. Now let us look at the same algebra but apply the diamond lemma with a different monomial ordering. First we order monomials by length and then lexicographically with $b > a > c$. This gives us the reductions:

$ba = ab$

$bc = cb$

This gives us no ambiguities and a basis for our algebra consisting of all monomials not containing the strings ba or bc. This is much simpler to use especially if we want to find the Hilbert series of this algebra.

We are interested in these instances where all ambiguities of degree three resolve (that is all ambiguities involving a monomial of length three resolve) not only because is it easier to compute the Hilbert series of such algebras. Once we have shown that there exists an ordering under the diamond lemma which causes ambiguities of degree three to resolve, we can use a theorem from [PP] to show our algebra is Koszul. We will need one definition and the following propositions, which are equivalent.

Definition 6. *If G is a graph with n nodes then a vertex ordering of G is a surjective map from the vertices of G onto $[n]$.*

Proposition 2.1. *Let A be a PGC-algebra with commuting graph $G_c(A)$. Suppose there exists a vertex ordering of $G_c(A)$ so for any three vertices a, b and c if $\{a, c\} \notin E, \{a, b\}, \{b, c\} \in E$ then neither $a < b < c$ nor $c < b < a$. Then there exists a monomial ordering so that all ambiguities of degree three are resolvable with the diamond lemma.*

Proposition 2.2. *Let A be a PGC-algebra with non-commuting graph $\overline{G_c(A)}$. Suppose there exists a vertex ordering of $\overline{G_c(A)}$ so for any three vertices a, b and c if $\{a, c\} \in E, \{a, b\}, \{b, c\} \notin E$ then neither $a < b < c$ nor $c < b < a$. Then there exists a monomial ordering so that all ambiguities of degree three are resolvable with the diamond lemma.*

Proof. Since the two statements are equivalent, we will prove only the first. Order monomials first by length and then by lexicographically extending the vertex ordering. It is enough to show that given any three distinct vertices a,b and c that all ambiguities involving those generators resolve.

First notice that if none of our generators a, b and c commute with each other, then there can be no ambiguity. The same holds if there is only one commuting pair.

If all three commute with each other, then we get one ambiguity which is resolvable since everything commutes.

Finally, consider the case where $\{a, c\} \notin E, \{a, b\}, \{b, c\} \in E$. The only ambiguities that could arise would come from the monomials abc or cba. However, since b is not in between c and a in the ordering it is not possible for both cb and ba to be reductions (and similarly ab and bc). Hence there is no ambiguity to resolve here and we are done. □

Example. For $\mathrm{ch}(P_n)$ we have the non-commuting graph as shown.

With this vertex ordering we have shown the algebra's ambiguities resolve. This also tells us (by the result in [PP]) that $\mathrm{ch}(P_n)$ is Koszul.

Now the propositions we have developed in this section only hold for PGC-algebras. However, now that we have an ordering of generators which worked for $\mathrm{ch}(P_n)$, we can try to use the same ordering for P_n and we will see that with this ordering all ambiguities still resolve. This will show P_n is Koszul, describe a basis for P_n, and show that P_n has the same Hilbert series as $\mathrm{ch}(P_n)$ implying that $\mathrm{ch}(P_n) = \mathrm{gr}(P_n)$.

Theorem 2. *P_n is Koszul.*

Proof. We show all ambiguities of degree three resolve. Our ordering is lexicographic with a generator ordering of $v_n > e_{n-1,n} > v_{n-1} > \ldots > e_{2,3} > v_2 > e_{1,2} > v_1$. Our reductions are

$v_k v_{k-1} = v_{k-1} v_k - e_{k,k-1} v_{k-1} + e_{k,k-1} v_k$

$v_k v_j = v_j v_k$ if $j < k - 1$

$v_k e_{k-1,k-2} = e_{k-1,k-2} v_k - e_{k,k-1} e_{k-1,k-2}$

$v_k e_{j,j-1} = ej, j - 1 v_k$ if $j < k - 1$

$e_{k,k-1} v_{k-2} = v_{k-2} e_{k,k-1} + e_{k-1,k-2} e_{k,k-1}$

$e_{k,k-1} v_j = v_j e_{k,k-1}$ if $j < k - 2$

$e_{k,k-1} e_{j,j-1} = e_{j,j-1} e_{k,k-1}$ if $j < k - 1$

Which gives us ambiguities of one these forms

$v_j v_k v_l$

$e_{j,j+1} v_k v_l$

$v_{j+1} e_{k,k+1} v_l$

$v_j v_k e_{l-1,l}$

$e_{j,j+1} v_k e_{l-1,l}$

for each of the four cases $j = k+1 = l+2, j > k+1 = l+2, j = k+1 > l+2, j > k+1 > l+2$ and

$e_{j,j+1}e_{k,k+1}v_l$ for $j-1 > k = l+1$ and $j-1 > k > l+1$

$v_j e_{k,k+1}e_{l,l+1}$ for $j-2 = k > l+1$ and $j-2 > k > l+1$

$e_{j,j+1}e_{k,k+1}e_{l,l+1}$ for $j-1 > k > l+1$.

This gives us 25 cases which need to be checked for P_n. The first n for which all 25 ambiguities actually appear is $n = 7$ and by symmetry it is enough for P_n to resolve the following ambiguities in L_7: $v_5v_3v_1, v_4v_3v_1, v_4v_2v_1, v_3v_2v_1,$ $v_4v_3e_{1,2}, v_5v_3e_{1,2}, v_5v_4e_{1,2}, v_6v_4e_{1,2},$ $v_4e_{3,2}v_1, v_5e_{3,4}v_1, v_5e_{2,3}v_1, v_6e_{3,4}v_1,$ $e_{3,4}v_2v_1, e_{4,5}v_2v_1, e_{4,5}v_3v_1, e_{5,6}v_3v_1, e_{4,5}v_3e_{1,2}, e_{5,6}v_3e_{1,2}, e_{5,6}v_4e_{1,2}, e_{6,7}v_4e_{1,2}$ $e_{4,5}e_{2,3}v_1, e_{5,6}e_{3,4}v_1,$ $v_5e_{3,4}e_{1,2}, v_6e_{3,4}e_{1,2},$ $e_{5,6}e_{3,4}e_{1,2}$. Each of these is easily checked. □

3 Duals and Hilbert Series

Now suppose A is any PGC-algebra with generators $a_1, a_2, \dots, a_n$. The relations will be $a_ia_j - a_ja_i$ for all $\{a_i, a_j\} \in G_c(A)$. Let b_i be the linear functional sending $b_i(a_i) = 1, b_i(a_j) = 0$ if $j \neq i$. Then the dual algebra A^* is given by generators $b_1, b_2, \dots, b_n$ and relations

$b_ib_j + b_jb_i$ for all $i \neq j, \{a_i, a_j\} \in G_c(A)$,

$b_ib_j = 0$ for all $i \neq j, \{a_i, a_j\} \notin G_c(A)$

${b_i}^2 = 0$ for all i

Notice that A^* is a quotient of the exterior algebra on $b_1, b_2, \dots, b_n$. A word in $T(span\{b_1, \dots, b_n\})$ is zero in A^* iff it contains two letters b_i and b_j so $\{a_i, a_j\} \notin G_c(A)$ (and hence b_ib_j=0 in A^*).

We can use this to compute the Hilbert series of P_n. Since we have already shown that ch(P_n) has the same Hilbert series as P_n we can work solely with ch(P_n) and use the fact that it is a PGC-algebra. In fact, since ch(P_n) is Koszul, our strategy will be to compute the Hilbert series H(x) of ch$(P_n)^*$ and get $\frac{1}{H(-x)}$ for the Hilbert series of P_n.

Remember that in ch(P_n) our generators were $\{v_1, \dots, v_n, e_{1,2}, \dots e_{n-1,n}\}$. Call the span of these generators V. The relations were simply that all the v_is commute with each other and $e_{i,i+1}$ commutes with everything except for $e_{i-1,i}, e_{i+1,i+2}, v_i,andv_{i+1}$. Let $w_i (i \in [n])$ be the functional $V \longrightarrow \mathbf{C}$ so that $w_i(v_i) = 1, w_i(v_j) = 0$ for $i \neq j$, and $w_i(e_{j,j+1}) = 0$. Let $d_{i,i+1}, i \in [n-1]$ be the functional so $d_{i,i+1}(e_{i,i+1}) = 1, d_{i,i+1}(e_{j,j+1}) = 0$ if $j \neq i$, and $d_{i,i+1}(v_j) = 0$. Then, by the reasoning we developed for general PGC-algebras, we get that A^* is the algebra given by generators $\{w_1, \dots, w_n, d_{1,2}, \dots d_{n-1,n}\}$ and relations

$d_{i,i+1}v_i = v_id_{i,i+1} = 0$

$d_{i,i+1}v_{i+1} = v_{i+1}d_{i,i+1} = 0$

$d_{i,i+1}d_{i+1,i+2} = d_{i+1,i+2}d_{i,i+1} = 0$

together with that all generators anti-commute. We will sometimes refer to the w_is as nodes and $d_{i,i+1}$s as edges, considering their origins.

Computing the Hilbert series of ch$(P_n)^*$ requires counting the number of subsets S of $\{w_1, \dots, w_n, d_{1,2}, \dots, d_{n-1,n}\}$ so that for each $d_{i,i+1} \in S$ we know $d_{i-1,i}, d_{i+1,i+2}, v_i, v_{i+1} \notin S$

Suppose S contains a total of j d's. Notice that since no two adjacent edges can be in S the number of ways we can have j d's is the number of matchings M(n,j) of the graph P_n of size j. Also, each edge rules out the possibility of exactly two vertices. Hence if $|S| = i$ then we have $\binom{n-2j}{i-j}$ ways we can pick the vertices for S. This gives us a total of M(n,j)$\binom{n-2j}{i-j}$ valid subsets containing j d's. The total number of valid subsets S, with $|S| = i$ is then $\sum_{j=0} M(n,j)\binom{n-2j}{i-j}$. This tells us that if we write the Hilbert series $H_n(x)$ of ch$(P_n)^*$ as $H_n(x) = H_n^0 + H_n^1x + H_n^2x^2 + \dots$ then we have $H_n^i = \sum_{j \geq 0} M(n,j)\binom{n-2j}{i-j}$. We set $H_n^i = 0$ in the cases where $n < 0$ or $i < 0$.

H_n^i is not the easiest thing to compute, so we will now find some rules to make finding the coefficients of $H_n(x)$ easier.

Proposition 3.1. *$H_n(x)$ is always a symmetric polynomial.*

Proof. We want to show $H_n^i = H_n^{n-i}$ or that $\sum_{j=0} M(n,j)\binom{n-2j}{i-j} = \sum_{j=0} M(n,j)\binom{n-2j}{n-k-j}$. But since $\binom{n-2j}{i-j} = \binom{n-2j}{n-2j-k+j} = \binom{n-2j}{n-j-k}$ we are done. □

If we write out a few $H_n(x)$ in a pyramid we notice that each term is the sum of the three terms in the triangle above it.

$$x^2 + 3x + 1$$
$$x^3 + 5x^2 + 5x + 1$$
$$x^4 + 7x^3 + 13x^2 + 7x + 1$$
$$x^5 + 9x^4 + 25x^3 + 25x^2 + 9x + 1$$

If we set H_n^k to be the coefficient of x^k in $H_n(x)$ we then get the following proposition.

Proposition 3.2. *For all $n \geq 2, k \geq 1$, $H_n^k = H_{n-1}^k + H_{n-1}^{k-1} + H_{n-2}^{k-1}$.*

Proof. We want

$$\sum_{j=0} M(n,j)\binom{n-2j}{k-j} =$$

$$\sum_{j=0} M(n-1,j)\binom{n-1-2j}{k-j} + \sum_{j=0} M(n-1,j)\binom{n-1-2j}{k-1-j} + \sum_{j=0} M(n-2,j)\binom{n-2-2j}{k-1-j}$$

or simply

$$\sum_{j=0}(M(n,j)\binom{n-2j}{k-j} - M(n-1,j)\binom{n-1-2j}{k-j} -$$
$$M(n-1,j)\binom{n-1-2j}{k-1-j} - M(n-2,j)\binom{n-2-2j}{k-1-j}) = 0$$

Notice the side we want to set to zero equals

$$\sum_{j=0} M(n,j)\binom{n-2j}{k-j} - M(n-1,j)(\binom{n-1-2j}{k-j} + \binom{n-1-2j}{k-1-j}) - M(n-2,j)\binom{n-2-2j}{k-1-j}$$

which by Pascal's identity is

$$\sum_{j=0} M(n,j)\binom{n-2j}{k-j} - M(n-1,j)\binom{n-2j}{k-j} - M(n-2,j)\binom{n-2-2j}{k-1-j}$$

which is just

$$\sum_{j=0}(M(n,j) - M(n-1,j))\binom{n-2j}{k-j} - M(n-2,j)\binom{n-2-2j}{k-1-j}$$

Notice also that a recurrence for the number of matchings of on the line P_n is given by $M(n,j) = M(n-1,j) + M(n-2,j-1)$. Using this on our first term gives us $\sum_{j=0}(M(n-2,j-1))\binom{n-2j}{k-j} - M(n-2,j)\binom{n-2-2j}{k-1-j}$ which collapses to zero and we are done. □

By adding another variable we get the following result explaining the coefficients of $H_n(x)$.

Proposition 3.3. $\sum_{n\geq 0} t^n H_n(x) = \frac{1}{1-t-xt-xt^2}$

Proof. We want to show

$$\sum_{n\geq 0}(1-t-xt-xt^2)t^n H_n(x) = 1$$

Since $H_n(x) = \sum_{k\geq 0} H_n^k x^k$ we have

$$\begin{aligned}
&\sum_{n\geq 0}(1-t-xt-xt^2)t^n H_n(x) \\
&= \sum_{n\geq 0}\sum_{k\geq 0}(1-t-xt-xt^2)H_n^k t^n x^k \\
&= \sum_{n,k\geq 0} H_n^k - tH_n^k - xtH_n^k - xt^2 H_n^k t^n x^k \\
&= \sum_{n,k\geq 0} H_n^k t^n x^k - \sum_{n,k\geq 0} H_n^k t^{n+1} x^k - \sum_{n,k\geq 0} H_n^k t^{n+1} x^{k+1} - \sum_{n,k\geq 0} H_n^k t^{n+2} x^{k+1} \\
&= \sum_{n,k\geq 0} H_n^k t^n x^k - \sum_{n,k\geq 0} H_{n-1}^k t^n x^k - \sum_{n,k\geq 0} H_{n-1}^{k-1} t^n x^k - \sum_{n,k\geq 0} H_{n-2}^{k-1} t^n x^k \\
&= \sum_{n,k\geq 0} (H_n^k - H_{n-1}^k - H_{n-1}^{k-1} - H_{n-2}^{k-1}) x^k t^n
\end{aligned}$$

We wish to show that this sum is equal to one. By our last proposition we know that $(H_n^k - H_{n-1}^k - H_{n-1}^{k-1} - H_{n-2}^{k-1}) = 0$ when $n \geq 2$ and $k \geq 1$. We break the sum into the three unknown cases of $k = 0$; $n = 0, k \geq 1$; and $n = 1, k \geq 1$ to get

$$\sum_{n\geq 0}(H_n^0 - H_{n-1}^0)t^n + \sum_{k\geq 1} H_0^k x^k + \sum_{k\geq 1}(H_1^k - H_0^k - H_0^{k-1})tx^k$$

For this to equal one we need to show:

i) $H_0^0 = 1$,

ii) $H_0^k = 0$ for $k \geq 1$,

iii) $H_1^k - H_0^k - H_0^{k-1} = 0$ for $k \geq 1$, and

iv) $H_n^0 - H_{n-1}^0 = 0$ when $n > 0$.

First $H_0^0 = \sum_{j\geq 0} M(0,j)\binom{0-2j}{0} = M(0,0)\binom{0}{0} = 1$

For ii) notice $H_0^k = \sum_{j=0} M(0,j)\binom{0-2j}{k-j} = M(0,0)\binom{0}{k} = 0$ because $k \geq 1$.

By ii), statement iii) now reduces to showing $H_1^1 - H_0^0 = 0$ and $H_1^k = 0$ for $k \geq 2$.

Notice $H_1^k = \sum_{j=0} M(1,j)\binom{1-2j}{k-j} = M(1,0)\binom{1}{k}$. Thus $H_1^k = 0$ for $k \geq 2$ and $H_1^1 = 1$. Since H_0^0 is one by i), this case is done.

Finally, for iv) notice $H_n^0 = \sum_{j\geq 0} M(n,j)\binom{n-2j}{-j} = M(n,0)\binom{n}{0} = 1$ and thus $H_n^0 - H_{n-1}^0 = 0$ when $n > 0$. □

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