An introduction to noncommutative algebra for non math majors

1 What is a set?

Before we can do any sort of Mathematics we need some starting point. Let's start by defining a set to be a collection of objects. We aren't going to limit ourselves to collections that are typically Mathematical. Colors, names, locations and other things can all be in our collections. What we would like, however, is to know for sure whether or not something is or is not in our set. In Mathematics, we need to be sure of what we are talking about in order to make claims about what is true or false. We are only going to talk about collections when we know what is actually in them. Often it is easy to see what constitutes a set and what doesn't. Sometimes it may not be as easy.

When we write a set down in a paper like this, we will list the objects of the set inside brackets whenever possible. We also will give usually give abbreviated names to our sets.

Example: Let us start with an easy example, the set of letters in the English language. I'll call this set A for alphabet. Notice that

$$A = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z\}.$$

This took a while to type, and might take a bit longer to write out by hand. In these sorts of situations we will often write something more like $\{A, B, C, ..., Z\}$ to save space and time. We will attempt to only do this in situations where what the ... represents is really obvious.

Example: Now lets think about the collection of words in the English language. Words can be viewed as objects, so at first thought this might make a good set. Unfortunately, there can be a lot of disagreement as to what constitutes a proper English word. Are we going by the Scrabble dictionary, the Webster's dictionary, or something else? Are slang words included? Curse words? What about old English words that aren't used anymore? Unless we specify what means we are using to tell if a word is acceptable, we can't really call this a set.

Example: Let us go back to the first collection of numbers you were introduced to, the set of nonnegative whole numbers. We will call this set $\mathbf{N}$.

Notice $\mathbf{N} = \{0, 1, 2, 3, ...\}$. Here we certainly don't want to try and write out each number, since we would never be able to finish the job.

You now know enough to move on to the next section. The next two examples are for braver readers who want to understand what problems might arise if we aren't careful with our rules and definitions regarding sets.

Example: (Not required and tricky) Sometimes it isn't easy to see when a description of a set is a good one. Let us for this example take the word number to mean an element of the set $\mathbf{N}$ we just defined. If we use the Webster's dictionary for our definition of what constitutes an English word, one could define a set D to be the set of all numbers describable in ten words or less. Some numbers that might fall into D would include seven, the first thirty digit number, and the sum of the first hundred prime numbers. These numbers are all in D since they are describable in ten words or less (taking one, five, and eight words respectively).

To be in this set you only need one description in ten words or less, though some numbers may have many. For instance, the number 1000 could be "one thousand" or "ten cubed" or "the square root of ten thousand". As long as you have one description under eleven words long you are in the set D.

Clearly not every number can be in the set D, since there are only finitely many possible sentences of ten words or less. Since there are infinitely many numbers in $\mathbf{N}$, there must be some number in $\mathbf{N}$ that is not in D.

Now consider the smallest number in $\mathbf{N}$ that is not in D. This would be "the smallest number not describable in ten words or less". It only took ten words to describe this number, so it must be in D. We have reached a contradiction with a number that is both in and not in D at the same time. Clearly something must be wrong. The problem arises from the fact that describability is not a rigorous property.

Example: (Not required and tricky) This next example is a bit more famous. Since a set is a collection of objects, why not consider the set where the objects are all of the sets. Since this collection of objects would be a set, this would be a set that contains itself.

Now that idea might seem a bit odd, but lets keep going with it and see what happens. Since this set contains itself, and we feel uncomfortable about that, lets consider the set U of all sets that do not contain themselves. This set seems a bit more normal at first, but isn't really.

Ask yourself if U is contained in the set U. If U is in U, then (since U is the set of all sets that do not contain themselves) U must not contain itself. Hence U is not in U. This sort of contradiction makes no sense, so suppose U is not in U. Then U does not contain itself and (since U is the set of all sets that do not contain themselves) U must be in U. We have another contradiction.

We've shown that if U is in U then U is not in U, and if U is not in U then U is in U. Something has to be wrong since this is ridiculous. It turns out that the set of all sets isn't a well defined notion of set either. In fact, the common axioms of mathematics allow no set to be a member of itself. We went wrong when we defined the set of all sets right in the second sentence.

There is an entire branch of Mathematics called Set Theory. It can be difficult at first seeing what we should allow to be a set, and what we shouldn't. The rest of the sets in the following sections bear no tricks and are all very simple to understand. In fact, we will for the most part stick to the set **N**.

2 Some Properties of Addition and Multiplication

If you are reading this you probably know a bit about addition. You have an idea of what it means to "add 5", but suppose first that I just told you to "add 5". You would need to know what it is I want you to add 5 to. So clearly addition is something that requires two numbers. Suppose I give you something to add 5 to, then after you performed the addition you would give me back one number. So we know that addition is an operation performed on two numbers with an output of one number. This is perhaps the most basic idea behind addition. The same holds true for multiplication. Our first description is the following: Multiplication and addition are operations taking two elements from a set to an element of the set.

We'll need some notation here to make things simpler, so we'll write $a \times b$ instead of "a times b" and $a + b$ instead of "a plus b" in most situations.

Now, for this section we could work with fractions or real numbers but let us stick to our set **N** of natural numbers for now to keep things as simple as possible.

Example: $7 \times 3 = 21 \quad 3 \times 7 = 21$

Example: $5 \times 2 = 10 \quad 2 \times 5 = 10$

Remember your multiplication table here? Most of us are required to memorize this in elementary school, and most of us developed some tricks for doing so. Once you learned that $7 \times 3 = 21$, you could use this to also multiply 3×7. This trick works for the whole table. What this means is that $a \times b = b \times a$ for all a and b in $\mathbf{N}$. This helpful property (when you get the same result if you switch the order of the two elements involved) is called commutativity. It also holds for addition.

Since this property works for more than one operation we already know, it might be a good idea to define it rigorously.

Definition: An operation $*$ on a set is commutative if $a * b = b * a$ for any elements a and b in our set.

There are many tricks for multiplication or addition that work in certain instances. I'm sure you can multiply anything by 1 or 10 very fast. You can probably add 0 quickly as well. Commutativity isn't like those tricks, because it always applies. It is a property that in some sense comes along with the operation.

Example: $(3 + 9) + 7 = 19 \quad (9 + 3) + 7) = 19$

Example: $(3 + 9) + 7 = 19 \quad 3 + (9 + 7) = 19$

Example: $(2 \times 3) \times 4 = 24 \quad 2 \times (3 \times 4) = 24$

In the first of these examples we can just use commutativity to see that $(3+9)+7$ is the same as $(9+3)+7$. We know the first term is equal in both calculations. However, knowing that $3+9 = 9+3$ isn't enough to help us see if $(3 + 9) + 7 = 3 + (9 + 7)$. In one situation we are asked to add the 3 and the 9 first, in the other we are asked to add the 9 and 7. Yet the two come out to be the same, and not just because of the particular values involved. It turns out that whenever we are adding three numbers, it doesn't matter whether we start on the first pair or the last pair. Regardless of the order in this sense, we get the same final result. This property is called associativity and it holds for both addition and multiplication.

Definition: An operation $*$ on a set is associative if $(a * b) * c = a * (b * c)$ for any elements a, b and c in our set.

Notice that so far, since both addition and multiplication have these properties, we don't have any good way to distinguish these two operations from each other. There does happen to be a property that connects (and distinguishes) these two operations called the distributive law and it says

$a \times (b+c) = (a \times b) + (a \times c)$. Notice that the law is not true if you switch the addition and multiplication since, for example, $1 + (0 \times 1) \neq (1+0) \times (1+1)$.

3 Some Properties not present in Addition and Multiplication

Coming soon.