ON THE FINITE-DIMENSIONAL IRREDUCIBLE
REPRESENTATIONS OF $PSL_2(\mathbb{Z})$

MELINDA G. MORAN AND MATTHEW J. THIBAULT

ABSTRACT. We classify up to equivalence all finite-dimensional irreducible representations of $PSL_2(\mathbb{Z})$ whose restriction to the commutator subgroup is diagonalizable.

1. Introduction

In this paper, we study finite-dimensional representations of the projective modular group $PSL_2(\mathbb{Z})$. This group is of fundamental importance in many fields including number theory, hyperbolic geometry, topological quantum field theory, and knot theory. Our main result completely classifies up to equivalence all finite-dimensional irreducible representations of $PSL_2(\mathbb{Z})$ whose restrictions to the commutator subgroup are diagonalizable. These representations are of dimension 1, 2, 3, and 6.

Recall that $PSL_2(\mathbb{Z})$ has the presentation $\langle x, y \mid x^2 = y^3 = 1 \rangle$. Its commutator subgroup has index 6 and is generated by the elements $xyxy^2, xy^2xy$; see [3]. Since the index is 6, it follows from standard Clifford Theory that the dimensions of the irreducible representations we are studying divide 6; see, for example, [2, 2.7] for further explanation. This fact is used in our analysis.

A complete classification of the finite-dimensional irreducible representations of $PSL_2(\mathbb{Z})$ with dimension $\leq 5$ follows from the work of Tuba and Wenzl [4] on representations of $B_3$. Let $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ be Artin’s braid group. Let $Z = \langle (\sigma_1\sigma_2)^3 \rangle$ denote the center of $B_3$. For appropriate choices of $A, B \in GL_n(k)$, $\sigma_1 \mapsto A$ and $\sigma_2 \mapsto B$ defines a representation of $B_3$. If $(AB)^3 = I$, then this also defines a representation of $B_3/Z \simeq PSL_2(\mathbb{Z})$. From the main theorem of Tuba and Wenzl [4], it follows that for $d \leq 5$, any $d$-dimensional irreducible representation of $B_3$ is uniquely determined by the eigenvalues of $A$ and the scalar by which $(AB)^3$ acts. Note that for any representation of $B_3$ induced from a representation of $PSL_2(\mathbb{Z})$, this scalar is 1. For each of the representations of $B_3$ induced by our 1-dimensional irreducible representations of $PSL_2(\mathbb{Z})$, the eigenvalue of $A$ is $\lambda$ where $\lambda^6 = 1$. If the characteristic of $k$ is not 3, then each of our 2-dimensional irreducible representations is described by the eigenvalues $\lambda, -\lambda$ where $\lambda^3 = 1$. If the characteristic of $k$ is not 2, then each of the two 3-dimensional irreducible representations we found is characterized by the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ or $\eta_1, \eta_2, \eta_3$ where $\lambda_i$ are the roots of $x^3 - 1 = 0$ and $\eta_i$ are the roots of $x^3 + 1 = 0$. Since each of our 6-dimensional irreducible representations of $PSL_2(\mathbb{Z})$ induce a representation of $B_3$ where the resulting matrix $A$ has eigenvalues $\lambda_i$ where $\lambda_i$ are the roots of $x^6 - 1 = 0$, we conclude that the 6-dimensional irreducible representations of $B_3$ are not uniquely determined by the eigenvalues of $A$ and the scalar by which $(AB)^3$ acts.
Adriaenssens and Le Bruyn [1] have recently analyzed specific families of representations of $\text{PSL}_2(\mathbb{Z})$ from the point of view of noncommutative algebraic geometry. In Proposition 5 of their paper, Adriaenssens and Le Bruyn describe a family of irreducible representations of $B_3$ which contains any sufficiently general irreducible representation of $B_3$. Indeed, included in this family are the representations of $B_3$ induced by the 1-dimensional, 2-dimensional, and 3-dimensional representations of $\text{PSL}_2(\mathbb{Z})$ that are characterized in this paper. Determining whether the representations of $B_3$ induced by our 6-dimensional representations are in this general family is computationally intensive and unknown. Subsequently constructed in [1] are two subfamilies of the general family of representations of $B_3$ described above. These are induced by representations of particular quotients of the path algebra of the quiver $Q$ associated with the formally smooth algebra $C_{\text{PSL}_2(\mathbb{Z})}$. The representations of $B_3$ induced by the 1-dimensional and 3-dimensional representations (but not the 2-dimensional representations) of $\text{PSL}_2(\mathbb{Z})$ detailed in this paper are included in the first subfamily. None of the representations of $B_3$ induced by the 1-dimensional, 2-dimensional, or 3-dimensional representations of $\text{PSL}_2(\mathbb{Z})$ studied in our paper are in the second subfamily. Again, determining whether the 6-dimensional representations of $B_3$ induced by our 6-dimensional representations are in these subfamilies requires further work.

The inspiration of this paper comes from the theory of highest weight modules. The finite-dimensional irreducible representations of a complex semisimple Lie algebra are diagonalizable over its Cartan subalgebra. In our study below, we view the commutator of $\text{PSL}_2(\mathbb{Z})$ as playing a role analogous to that of the Cartan subalgebra.

Another way to view our work is as follows: Set $G'$ as the commutator subgroup of $G$ and $G''$ as the second commutator subgroup of $G$. It is well known (see [3]) that $G'$ is a free group in the two generators $xyxy^2$ and $xy^2xy$, and so $G'/G''$ is a free abelian group on two generators. In particular, the irreducible representations of $G'/G''$ are all one-dimensional. Therefore, by Clifford theory, the restrictions to $G'/G''$ of irreducible finite-dimensional representations of $G/G''$ must be diagonalizable. Conversely, every finite-dimensional irreducible representation of $G/G''$ lifts to a representation of $G$ whose restriction to $G'$ naturally factors through $G/G''$. We see that our results amount to a complete classification, up to equivalence, of the irreducible finite-dimensional representations of $G/G''$. Moreover, since $G/G''$ is abelian-by-finite, the irreducible representations are all finite-dimensional.

Our approach is mostly elementary, relying on basic linear algebraic computations and case-by-case analyses. Toward the end of the paper, extensive Maple calculations are used to determine the 6-dimensional representations; an appendix of the Maple code used is included.

Our main results are stated in section 4. Sections 5 and 6 are devoted to the proofs. Preliminary results and notation are given in sections 2 and 3.

Acknowledgements: This work was begun at the Temple University 2004 Research Experience for Undergraduates, funded by NSF REU Site Grant DMS-0138991. We would like to thank our REU supervisor Professor E. Letzter for formulating the question, for his advice on this research project, and for his help in writing this paper. We are also happy to acknowledge helpful discussion of this work with D. Vogan.
2. Definitions and Notation

We begin with some relevant definitions. Let $k$ denote an algebraically closed field and $M_n(k)$ denote the set of all $n \times n$ matrices with entries in $k$. Let $GL_n(k)$ denote the set of all invertible elements of $M_n(k)$.

Definition 2.1. We will say the ordered $m$-tuple $(A_1, A_2, \ldots, A_m)$, $A_i \in M_n(k)$ for $i = 1, 2, \ldots, m$, is irreducible if every element of $M_n(k)$ can be written as a $k$-linear combination of products in the $A_i$’s, $i = 1, 2, \ldots, m$.

Also, if $X_i, X_i' \in M_n(k)$ for $i = 1, 2, \ldots, m$, then $(X_1', X_2', \ldots, X_m')$ is equivalent to $(X_1, X_2, \ldots, X_m)$, denoted $(X_1', X_2', \ldots, X_m') \approx (X_1, X_2, \ldots, X_m)$, if there exists $Q \in GL_n(k)$ such that $X_i' = QX_iQ^{-1}$ for all $i = 1, 2, \ldots, m$.

Notation 2.2. The following notation will remain in effect for the entire paper: Let $G$ denote the set of all invertible elements of $M_n(k)$.

Remark 3.1. Since $k$ is algebraically closed, all irreducible solutions to $XY = YX$ in $M_n(k)$ are one-dimensional.

Lemma 3.2. All irreducible representations of $PSL_2(\mathbb{Z})$ with $\Lambda = \Gamma$, or $\Lambda$ or $\Gamma$ a scalar matrix are one-dimensional.

Proof. If $\Lambda = \Gamma$, then $XY = Y^2XY^2 = YX$. For an arbitrary constant $c$, computation with $\Lambda = cI$ yields $XY = (Y^2X)^2(YX) = (Y^2X)^2\Lambda(YX) = YX$. Similarly, computation with $\Gamma = cI$ yields $XY = (Y^2X)(YX)\Gamma(Y^2XY^2) = (Y^2X)\Gamma(YX)(Y^2XY^2) = YX$.

Remark 3.1 thus concludes this proof. □

Remark 3.3. Assume $\Lambda$ and $\Gamma$ are $n \times n$ diagonal matrices, where $\Lambda_{i,i} =: \lambda_i$ and $\Gamma_{i,i} =: \gamma_i$ for $i = 1, 2, \ldots, n$. We observe the following properties of $\Lambda$ and $\Gamma$:

- $\Lambda X \Lambda = X$, $\Gamma X \Gamma = X$, $\Lambda Y \Gamma = Y$, $\Gamma Y^2 \Lambda = Y^2$
- $\Lambda \Gamma = \Gamma \Lambda$, so $(XY)^6 = (YX)^6 = I$.

- Since $Y$ has at least one nonzero entry per row, $\Lambda$ and $\Gamma$ are conjugate, and $\Lambda Y \Gamma = Y$, then $\frac{\lambda_i}{\lambda_j} = \gamma_j = \lambda_k$ for each $i = 1, 2, \ldots, n$, and some $j, k \in \{1, 2, \ldots, n\}$. Note that $\Lambda, \Gamma \in GL_n(k)$, so $\lambda_i, \gamma_i \neq 0$ for each $i = 1, 2, \ldots, n$.

The proof of the following lemma is routine and omitted.

Lemma 3.4. Let the following properties hold:

1. $X, Y \in GL_n(k)$ and have exactly one nonzero entry per row and column
(2) \((X,Y)\) is irreducible

(3) \(X^2 = Y^3 = I\)

(4) \(\Lambda, \Gamma\) are diagonal matrices

Then \((X,Y)\) satisfies the above properties if and only if \((PXP^{-1}, PYP^{-1})\) satisfies the above properties, where \(P\) is a non-singular weighted permutation matrix (i.e. \(P\) has exactly one nonzero entry per row and column).

4. Main Results

Let \(\rho\) be an irreducible representation of \(PSL_2(\mathbb{Z})\) which maps the commutator subgroup of \(PSL_2(\mathbb{Z})\) to diagonal matrices in \(GL_n(k)\). Since the index of the commutator subgroup is 6, it follows from standard Clifford Theory (see, e.g., [2,2.7] that the dimension of \(\rho\) divides 6. We thus analyze the cases when \(n = 1, 2, 3,\) and 6.

**Theorem 4.1.** Let \(k\) be an algebraically closed field, and let \(\zeta\) be a primitive cube root of unity if \(k\) is not of characteristic 3. Let \(\rho : PSL_2(\mathbb{Z}) \to GL_n(k)\) be an irreducible representation of \(PSL_2(\mathbb{Z}) = \langle x, y \mid x^2 = y^3 = 1 \rangle\) which maps the commutator subgroup of \(PSL_2(\mathbb{Z})\) to diagonal matrices in \(GL_n(k)\).

i) If \(k\) is not of characteristic 2 or 3, then \((\rho(x), \rho(y))\) is equivalent to one of the following:

\[ (1, 1), (-1, 1), (1, \zeta), (-1, \zeta), (1, \zeta^2), (-1, \zeta^2) \]

\[ (2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}, (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \zeta^2 \end{pmatrix}), (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}) \]

\[ (3, \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \]

\[ (4, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ c_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \end{pmatrix}) \]

for \(c_1, c_2 \in k, c_1, c_2 \neq 0, (c_1, c_2) \neq (1, 1), (-1, 1), (1, -1), (-1, -1), (\zeta, \zeta), (\zeta^2, \zeta^2).\)

ii) If \(k\) is of characteristic 2, then \((\rho(x), \rho(y))\) is equivalent to one of the following:

\[ (1, 1), (1, \zeta), (1, \zeta^2) \]

\[ (2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}, (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \zeta^2 \end{pmatrix}), (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}) \]
Remark 4.2. The remainder of the paper is devoted to the proof of this theorem.

In this section, we prove (i), (ii), and (iii) of Theorem 4.1, considering separately the cases when \( n = 1, 2, \) and 3.

5. Cases when \( n=1, n=2, \) and \( n=3 \)

In this section, we prove (i), (ii), and (iii) of Theorem 4.1, considering separately the cases when \( n = 1, 2, \) and 3.

5.1. \( n=1 \). For \( n = 1 \), \( \Lambda \) and \( \Gamma \) are trivially diagonal. We only require that \( (X,Y) = (a,b) \) where \( a^3 = b^3 \).

Distinct \((X,Y)\) create distinct equivalence classes since \((X',Y') \approx (X,Y)\) requires that \((X',Y') = (QXQ^{-1},QYQ^{-1}) = (X,Y)\). In fields of characteristic 2, the only solution to \( X^2 = 1 \) is \( X = 1 \). In fields of characteristic other than 2, \( X^2 = 1 \) has the 2 distinct solutions \( X = \pm 1 \). In fields of characteristic 3, the only solution to \( Y^3 = 1 \) is \( Y = 1 \). In fields of characteristic other than 3, \( Y^3 = 1 \) has the 3 distinct solutions \( Y = 1, \zeta, \zeta^2 \). We therefore arrive at the desired result.
5.2. **n=2.** Let $\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}$. First assume $\lambda_1 = \lambda_2$. Then $\Lambda = \lambda_1 I$, which is a contradiction by Lemma 3.2. Thus assume $\lambda_1 \neq \lambda_2$. By explicitly solving $\Lambda X \Lambda = X$, we find that for any $i, j \in \{1, 2\}$, $\lambda_i \lambda_j = 1$ if $X_{i,j} \neq 0$. If $X$ has more than one nonzero entry per row or column, it follows that $\lambda_1 = \lambda_2$, which is a contradiction to our assumption. Therefore since $X$ is non-singular, $X$ has exactly one nonzero entry per row and column.

Since $\Lambda$ and $\Gamma$ are conjugate, we find $\Gamma$ has diagonal entries $\gamma_1$, $\gamma_2$ where $\gamma_1 \neq \gamma_2$. Solving $\Lambda Y \Gamma = Y$, we find that for any $i, j \in \{1, 2\}$, $\lambda_i \gamma_j = 1$ if $Y_{i,j} \neq 0$. If $Y$ has more than one nonzero entry per row or column, then either $\lambda_1 = \lambda_2$ or $\gamma_1 = \gamma_2$, both of which are contradictions. Therefore since $Y$ is non-singular, $Y$ has exactly one nonzero entry per row and column. Hence $X$ and $Y$ are of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ or $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. If $Y$ is of the latter type, then $Y^3 \neq I$, so $Y$ must be diagonal. Clearly $X$ is of the latter form by the irreducibility of $(X, Y)$. Using $X^2 = I$, we see that $X = \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix}$ for some nonzero $x_1 \in k$. Conjugating $X$ and $Y$ by the weighted permutation matrix $P = \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix}$, we see by Lemma 3.4 that

$(X, Y) \approx (X_1, Y_1) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $(X_1, Y_1)$ satisfies the hypotheses of Lemma 3.4. By Remark 3.1, $a \neq b$. Also, if $a \neq b$,

$$\frac{1}{a-b}(X_1 Y_1 - b X_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \frac{1}{a-b}(a X_1^2 - Y_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{1}{a-b}(a X_1 - X_1 Y_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \frac{1}{a-b}(Y_1 - b X_1^2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$  

Thus $(X_1, Y_1)$ is irreducible if and only if $a \neq b$.

Finally, $a^3 = b^3 = 1$ by $Y_1^3 = I$. In fields of characteristic 3, $a = b = 1$ since $a^3 = b^3 = 1$, which is a contradiction. Thus there are no irreducible representations from $PSL_2(\mathbb{Z})$ to $GL_2(k)$ if $k$ has characteristic 3. In fields not of characteristic 3, we have

$$(a, b) = (1, \zeta), (1, \zeta^2), (\zeta, \zeta^2), (\zeta, 1), (\zeta^2, 1), \text{ or } (\zeta^2, \zeta).$$

Conjugating $X_1$ and $Y_1$ by permutation matrix $X_1$ yields $(X_1, Y_1) \approx (X_1, \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix})$. Thus we may assume the following cases: $(a, b) = (1, \zeta), (1, \zeta^2), \text{ or } (\zeta, \zeta^2)$. Since $tr(Y_1)$ is distinct for each of the listed cases, each yields a separate equivalence class. This gives the desired result.

5.3. **n=3.** Let $\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2 & \lambda_3 & \lambda_1 \\ \lambda_3 & \lambda_1 & \lambda_2 \end{pmatrix}$ and $\Gamma = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_1 \\ \gamma_3 & \gamma_1 & \gamma_2 \end{pmatrix}$. By Lemma 3.4, we may assume that either $\lambda_1 = \lambda_2 = \lambda_3$, $\lambda_1 = \lambda_2 \neq \lambda_3$, or that $\lambda_1$, $\lambda_2$, and $\lambda_3$ are distinct.

In the first case $\Lambda = \lambda_1 I$, which is a contradiction by Lemma 3.2.

We will now solve the third case. By solving $\Lambda X \Lambda = X$, we conclude for any $i, j \in \{1, 2, 3\}$ that $\lambda_i \lambda_j = 1$ if $X_{i,j} \neq 0$. If $X$ has more than one nonzero entry per row or column, we find that $\lambda_i = \lambda_j$ for some $i \neq j$, which is a contradiction to our
assumption that \( \lambda_1, \lambda_2, \lambda_3 \) are distinct. Therefore since \( X \) is non-singular, \( X \) has exactly one nonzero entry per row and column. Solving \( \Lambda Y \Gamma = Y \), we find that for any \( i, j \in \{1, 2, 3\} \), \( \lambda_i \gamma_j = 1 \) if \( Y_{i,j} \neq 0 \). If \( Y \) has more than one nonzero entry per row or column, then either \( \lambda_i = \lambda_j \) or \( \gamma_i = \gamma_j \) for some \( i \neq j \), both of which are contradictions. Therefore since \( Y \) is non-singular, \( Y \) has exactly one nonzero entry per row and column. Note, \( \lambda_k = \frac{1}{\lambda_i} \) for some \( i = 1, 2, 3 \) and each \( k \), by Remark 3.3. Since there are at most two distinct solutions to \( x = \frac{1}{2} \), then \( \lambda_i \neq \frac{1}{\lambda_i} \) for some \( i \). Conjugating \( X \) and \( Y \) by a permutation matrix if necessary, we may assume that \( \lambda_2 = \frac{1}{\lambda_1} \). Thus \( \lambda_3 = \frac{1}{\lambda_2} \). Therefore, we find \( \lambda_3 \lambda_1 \neq 1 \) and \( \lambda_3 \lambda_2 \neq 1 \). Hence \( X_{3,1} = X_{3,2} = 0 \). This forces \( X_{3,3} \neq 0 \). Expanding \( \Gamma X \Gamma = X \), we see from \( X_{3,3} \neq 0 \) that \( \gamma_3 = \frac{1}{\lambda_3} \). Because \( \Lambda \) and \( \Gamma \) are conjugate, \( \gamma_3 = \lambda_3 = \frac{1}{\lambda_3} \). Since \( \gamma_3 = \frac{1}{\lambda_3} \) and \( \gamma_1, \gamma_2, \gamma_3 \) are distinct, we find \( \lambda_2 \gamma_2 \neq 1 \) and \( \lambda_3 \gamma_1 \neq 1 \). Thus \( Y_{3,1} = Y_{3,2} = 0 \). This forces \( Y_{3,3} \neq 0 \). But since \( X \) and \( Y \) have exactly one nonzero entry per row and column and \( X_{3,3} \) and \( Y_{3,3} \) are nonzero, we find that \( (X, Y) \) is reducible, which is a contradiction.

In case 2, we assume \( \lambda_1 = \lambda_2 \neq \lambda_3 \). First assuming \( \lambda_1 \neq \frac{1}{\lambda_1} \) and using Remark 3.3, we see \( \lambda_3 = \frac{1}{\lambda_2} \). Again solving \( \Lambda X \Lambda = X \), we conclude for any \( i, j \in \{1, 2, 3\} \) that \( \lambda_i \lambda_j = 1 \) if \( X_{i,j} \neq 0 \). From \( \lambda_3 = \frac{1}{\lambda_2} \) and \( \lambda_1 = \lambda_2 \neq \lambda_3 \), we see that \( \lambda_1^2 = \lambda_1 \lambda_2 = \lambda_2 \lambda_1 = \lambda_2^2 \neq \lambda_1 \lambda_3 = 1 \). Then \( X_{1,1} = X_{1,2} = X_{2,1} = X_{2,2} = 0 \). But this implies that \( X \) is singular, which is a contradiction. Hence \( \lambda_1 = \frac{1}{\lambda_1} \) and \( \lambda_3 = \frac{1}{\lambda_3} \).

In fields of characteristic 2, there is only one distinct solution to \( x = \frac{1}{2} \). This is a contradiction since \( \lambda_1 \neq \lambda_3 \). Hence if \( k \) has characteristic 2, there are no irreducible representations from \( PSL_2(\mathbb{Z}) \) to \( GL_3(k) \).

We now consider fields \( k \) which are not of characteristic 2. Since \( \lambda_i = \frac{1}{\lambda_i} \) for \( i = 1, 2, 3 \), then \( \Lambda = \pm \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \). Because \( \Lambda \) and \( \Gamma \) are conjugate and \( \Lambda \neq \Gamma \), we have 4 possibilities for \( (\Lambda, \Gamma) \). Substituting each of the possibilities for \( (\Lambda, \Gamma) \) in \( \Lambda X \Lambda = X = \Gamma X \), we solve the resulting system of equations to find that in each possibility, \( X \) is diagonal. Again substituting each of the possibilities for \( (\Lambda, \Gamma) \) in the equation \( \Lambda Y \Gamma = Y \) and using \( Y^3 = I \), we solve the resulting system of equations. In each possibility, \( Y \) has exactly one nonzero entry per row and column.

Since \( Y \) has exactly one nonzero entry per row and column, \( Y^3 = I \), \( X \) is diagonal, and \( (X, Y) \) is irreducible, we conclude \( Y = \begin{pmatrix} 0 & y_1 & 0 \\ 0 & 0 & y_2 \\ y_3 & 0 & 0 \end{pmatrix} \) or \( \begin{pmatrix} 0 & 0 & y_1 \\ y_2 & 0 & 0 \\ 0 & y_3 & 0 \end{pmatrix} \), where \( y_1 y_2 y_3 = 1 \). Conjugate \( (X, Y) \) with \( P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & y_1 \\ 0 & y_1 y_3 & 0 \end{pmatrix} \) in case 1, or with \( P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & y_1 \\ 0 & y_1 y_3 & 0 \end{pmatrix} \) in case 2, so that \( (X, Y) \approx (P X P^{-1}, P Y P^{-1}) =: (X_1, Y_1) \), where \( X_1 \) is diagonal and \( Y_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \). Since \( X_1^2 = I \), all of the diagonal entries must be \( \pm 1 \). Because \( X_1 \neq \pm I \), there are 6 possibilities for \( X_1 \). Conjugating
by various permutation matrices, we can see that

$$(X, Y) \approx (\pm X_2, Y_1), \text{ where } X_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Thus $(X, Y)$ satisfies the hypotheses of Lemma 3.4 if and only if $(\pm X_2, Y_1)$ satisfies the hypotheses of Lemma 3.4. Since $(X_2, Y_1)$ satisfies $X_2^2 = I = Y_1^3$ and $X_2Y_1X_2Y_1^2$ and $X_2Y_1^2X_2Y_1$ are diagonal, we need only check irreducibility.

If $(X_2, Y_1)$ is irreducible, then $(-X_2, Y_1)$ is irreducible. Hence it is sufficient to check irreducibility for $(X_2, Y_1)$. However, we note that the standard basis matrices can be composed as:

$$Y_1 - Y_1X_2, \frac{Y_1(Y_1 - Y_1X_2)}{2}, \frac{Y_1^2(Y_1 - Y_1X_2)}{2}, Y_1(Y_1 - Y_1X_2)Y_1, \frac{Y_1(Y_1 - Y_1X_2)Y_1^2}{2}, \frac{Y_1^2(Y_1 - Y_1X_2)Y_1^2}{2}, \frac{Y_1^2(Y_1 - Y_1X_2)Y_1^2}{2}, \frac{Y_1^2(Y_1 - Y_1X_2)Y_1^2}{2}.$$

Note that $(X_2, Y_1)$ and $(-X_2, Y_1)$ yield separate equivalence classes since the trace of the matrices are preserved in each equivalence class and $tr(X_2) \neq tr(-X_2)$. We have therefore achieved the desired result.

6. Case when $n=6$

In this section, we prove (i), (ii), and (iii) of Theorem 4.1 in the case when $n = 6$.

**Lemma 6.1.** Suppose $X, Y \in GL_n(k)$ where $X^2 = Y^3 = I$, $(X, Y)$ is irreducible, and $\Lambda = XYXY^2$ and $\Gamma = XY^2XY$ are diagonal matrices. Then $X$ and $Y$ must have exactly one nonzero entry per row and column.

**Proof.** Assume $X$ and $Y$ satisfy the hypothesis of the lemma. Let $\Lambda$ and $\Gamma$ be diagonal matrices with diagonal entries $\Lambda_{i,i} = \lambda_i, \Gamma_{i,i} = \gamma_i$. Note that $(X, Y)$ satisfies the hypothesis if and only if $(PX^{-1}, PY^{-1})$ satisfies the hypothesis where $P$ is a permutation matrix. Thus we may assume the diagonal entries of $\Lambda$ are in one of the following cases, where entries in distinct ordered tuples are not equal and entries in the same ordered tuple are equal:

1. $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$
2. $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$, $(\lambda_6)$
3. $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $(\lambda_5, \lambda_6)$
4. $(\lambda_1, \lambda_2, \lambda_3)$, $(\lambda_4, \lambda_5, \lambda_6)$
5. $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $(\lambda_5)$, $(\lambda_6)$
6. $(\lambda_1, \lambda_2, \lambda_3)$, $(\lambda_4, \lambda_5)$, $(\lambda_6)$
7. $(\lambda_1, \lambda_2)$, $(\lambda_3, \lambda_4)$, $(\lambda_5, \lambda_6)$
8. $(\lambda_1, \lambda_2, \lambda_3)$, $(\lambda_4)$, $(\lambda_5)$, $(\lambda_6)$
9. $(\lambda_1, \lambda_2)$, $(\lambda_3, \lambda_4)$, $(\lambda_5)$, $(\lambda_6)$
10. $(\lambda_1, \lambda_2)$, $(\lambda_3)$, $(\lambda_4)$, $(\lambda_5)$, $(\lambda_6)$
11. $(\lambda_1)$, $(\lambda_2)$, $(\lambda_3)$, $(\lambda_4)$, $(\lambda_5)$, $(\lambda_6)$

**Case 1:**
In this case $\Lambda = \lambda_1 I$, which by Lemma 3.2 is a contradiction to the irreducibility of $(X, Y)$.

**Case 2, 3:**
First assume $\lambda_1 \neq \frac{1}{\lambda_i}$. Expanding $\Lambda X \Lambda = X$ we get a system of equations where
for any \(i, j \in \{1, \ldots, 6\}\), \(\lambda_i \lambda_j = 1\) if \(X_{i,j} \neq 0\). Since \(\lambda_1 \neq \frac{1}{\lambda_1}\), we find \(X\) must be singular, which is a contradiction. Thus assume \(\lambda_1 = \frac{1}{\lambda_1}\). We conclude from Remark 3.3 that \(\lambda_6 = \frac{1}{\lambda_6}\) and hence \(\Lambda^2 = I\). Because \(\Lambda \Gamma = \Gamma \Lambda\), we see that \((XY)^6 = (YX)^6 = I\). Also, \((XY)^3 = (YX)^3\) since \(\Lambda^2 = I\).

Let \(A = XY\) and \(B = YX\). Since \(A^3 = B^3\), \(ABA = BAB\), and \(A^6 = B^6 = I\), we can count possible monomials in \(A\) and \(B\) to find that \((A, B)\) span at most a 24-dimensional space and thus cannot be irreducible. Note \((A, B)\) is irreducible if and only if \((X, Y)\) is irreducible, since we can generate \(X\) and \(Y\) from \(A\) and \(B\) and vice versa. Therefore \((X, Y)\) is not irreducible.

Case 4:
By Remark 3.3, if \(\lambda_1 = \frac{1}{\lambda_1}\), then \(\lambda_6 = \frac{1}{\lambda_6}\) and \(\Lambda^2 = I\). As in Cases 2 and 3, this leads to a contradiction. Hence \(\lambda_6 = \frac{1}{\lambda_6}\). Observe that since \(\Lambda\) and \(\Gamma\) are conjugate,

\[
(6.1) \quad \Lambda + \Lambda^{-1} = (\lambda_1 + \frac{1}{\lambda_1})I = \Gamma + \Gamma^{-1}.
\]

Then it follows that

\[
(6.2) \quad Y(\Lambda + \Lambda^{-1}) = Y(\Gamma + \Gamma^{-1}) = (\Gamma + \Gamma^{-1})Y.
\]

Substituting \(\Lambda = XYXY^2\) and \(\Gamma = XY^2XY\) into equation (6.2) and using \((XY)^6 = I\) yields

\[
(6.3) \quad (XY)((XY)^2 - (YX)^2)(XY - YX) = 0.
\]

Since \(XY = YX\Gamma\), we find that \((XY)((XY)^2 - (YX)^2)(\Gamma - I) = 0\). Given that \(\Lambda\) and \(\Gamma\) are conjugate and \(\lambda_1\) and \(\lambda_6\) are not equal to 1, \((\Gamma - I)\) is invertible. Also by invertibility of \((XY)\), we deduce \((XY)^2 = (YX)^2\). Thus \(A^2 = B^2\), \(A^6 = B^6 = I\), and \(ABA = BAB\), where \(A = XY\) and \(B = YX\). Again by counting monomials in \(A\) and \(B\), we find \(A\) and \(B\) span at most an 18-dimensional space and thus \((A, B)\) cannot be irreducible. Since \((A, B)\) is irreducible if and only if \((X, Y)\) is irreducible, \((X, Y)\) cannot be irreducible.

Case 5:
First assume \(\lambda_1 \neq \frac{1}{\lambda_1}\). Then by Remark 3.3, \(\lambda_5\) or \(\lambda_6\) must equal \(\frac{1}{\lambda_5}\) or \(\frac{1}{\lambda_6}\). We expand \(\Lambda X \Lambda = X\) to get a system of equations where for any \(i, j \in \{1, \ldots, 6\}\), \(\lambda_i \lambda_j = 1\) if \(X_{i,j} \neq 0\). If either \(\lambda_5 = \frac{1}{\lambda_5}\) or \(\lambda_6 = \frac{1}{\lambda_6}\), then \(X\) must be singular, which is a contradiction. Thus \(\lambda_1 = \frac{1}{\lambda_1}\). If \(\lambda_5 = \frac{1}{\lambda_5}\) then by Remark 3.3, \(\lambda_6 = \frac{1}{\lambda_6}\). This is a contradiction since there are at most two distinct solutions to \(x = \frac{1}{x}\). Hence \(\lambda_5 = \frac{1}{\lambda_5}\). Since \(\Lambda\) and \(\Gamma\) are conjugate, there are \(6, 4, 1, 1\) permutations of the \(\lambda_i\)’s. Thus there are 30 possible matrices for \(\Gamma\) with nonzero entries determined by the \(\lambda_i\)’s. By case-by-case checking, we see that the restrictions \(\Gamma X \Gamma = X = \Lambda X \Lambda\) and \(X\) is non-singular leave 14 possible matrices for \(\Gamma\) (see Appendix 1). Note that \(XY = YX\) if \(\Lambda = \Gamma\) and \((XY)^2 = (YX)^2\) if \(\Lambda \Gamma = I\). In both cases, we find \((X, Y)\) is not irreducible as shown earlier. This leaves 12 possible matrices for \(\Gamma\).

Let \(P\) be a permutation matrix. Note that \((X, Y)\) satisfies the hypothesis of Lemma 6.1 if and only if \((PX P^{-1}, PY P^{-1})\) does. Also, \(X\) and \(Y\) have one nonzero entry per row and column if and only if \(PX P^{-1}\) and \(PY P^{-1}\) do. Thus for any permutation matrix \(P\), we may replace \((X, Y)\) with \((PX P^{-1}, PY P^{-1})\). Further
case-by-case checking gives us that for each of the 12 possible values of \( \Gamma \), we are able to replace \((X, Y)\) with \((PXP^{-1}, PYP^{-1})\) for an appropriate permutation matrix \( P \), so that \( \Lambda \) is preserved, \( \gamma = \gamma_6 = \gamma_5 = \lambda_1, \gamma_3 = \lambda_5, \) and \( \gamma_5 = \lambda_6 \) (see Appendix 1). Substituting \( \Lambda \) and \( \Gamma \) into \( \Lambda X \Lambda = Y \) and \( \Gamma Y^2 \Lambda = Y^2 \), using \( Y_3 = I \), and solving for \( Y \), we find that \( Y = \begin{pmatrix} 0 & 0 & Y_1 \\ Y_2 & 0 & 0 \\ 0 & Y_3 & 0 \end{pmatrix} \), where \( Y_1, Y_2, \) and \( Y_3 \) are 2 \( \times \) 2 block matrices. Similarly, by substituting \( \Lambda \) and each possible \( \Gamma \) into \( \Lambda X \Lambda = X = \Gamma X \Gamma \) and solving for \( X \), we find that \( X = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix} \), where \( X_1, X_2, \) and \( X_3 \) are 2 \( \times \) 2 block matrices.

Due to the fact that \( Y^3 = I \), we find \( Y_3 = Y_1^{-1}Y_2^{-1} \). Computation with \( XY = 
\Lambda Y \Lambda Y \) yields

\[
X_1Y_1 = \lambda_1Y_1X_3 \quad X_2Y_2 = \lambda_1Y_2X_1 \quad X_3Y_3 = \begin{pmatrix} \lambda_5 & 0 \\ 0 & \frac{1}{X_3} \end{pmatrix} Y_3X_2.
\]

But then:

\[
X_3Y_3 = \frac{1}{\lambda_1}(Y_1^{-1}X_1Y_1)Y_3 = \frac{1}{\lambda_1}(Y_1^{-1})(X_1Y_2^{-1}) = \frac{1}{\lambda_1}(Y_1^{-1})(\frac{1}{\lambda_1})(Y_2^{-1}X_2) = \left(\frac{1}{\lambda_1}\right)Y_3X_2.
\]

This is a contradiction to \( X_3Y_3 = \begin{pmatrix} \lambda_5 & 0 \\ 0 & \frac{1}{X_3} \end{pmatrix} Y_3X_2 \).

Case 6:
Suppose \( \lambda_6 \neq \frac{1}{X_3} \). Then either \( \lambda_1 = \frac{1}{X_6} \) or \( \lambda_4 = \frac{1}{X_6} \). We expand \( \Lambda X \Lambda = X \) to get a system of equations where for any \( i, j \in \{1, \ldots, 6\}, \lambda_i \lambda_j = 1 \) if \( X_{i,j} \neq 0 \). If either \( \lambda_1 = \frac{1}{X_6} \) or \( \lambda_4 = \frac{1}{X_6} \), then \( X \) must be singular, which is a contradiction. Thus \( \lambda_6 = \frac{1}{X_3} \). Suppose next that \( \lambda_4 \neq \frac{1}{X_6} \). Then \( \lambda_1 = \frac{1}{X_4} \). Since \( \lambda_i \lambda_j = 1 \) if \( X_{i,j} \neq 0 \) and since \( \lambda_1 = \frac{1}{X_4} \), we find that \( X \) is singular, which is a contradiction. Thus assume \( \lambda_4 = \frac{1}{X_3} \) and \( \lambda_6 \neq \frac{1}{X_3} \). It follows that \( \lambda_1 = \frac{1}{X_3} \). This is a contradiction since there are at most 2 distinct solutions to \( x = \frac{1}{x} \), while \( \lambda_1, \lambda_4, \) and \( \lambda_6 \) are distinct.

Case 7:
It is easy to show using Remark 3.3 that there must be an even number of ordered tuples with entries \( \lambda_i \) such that \( \lambda_i \neq \frac{1}{X_3} \). Since there are 3 tuples in this case, there must be 1 or 3 tuples with entries \( \lambda_i \) where \( \lambda_i = \frac{1}{X_3} \). Because there are at most two distinct solutions to \( x = \frac{1}{x} \), there is only one tuple with entries \( \lambda_i \) where \( \lambda_i = \frac{1}{X_3} \). Without loss of generality, assume that \( \lambda_1 = \frac{1}{X_3} \). Therefore \( \lambda_3 = \frac{1}{X_3} \). Since \( \Lambda \) and \( \Gamma \) are conjugate, there are \( \begin{pmatrix} 6 \\ 2, 2, 2 \end{pmatrix} \) = 90 possible matrices for \( \Gamma \) in terms of the \( \lambda_i \)'s. From \( \Gamma X \Gamma = X = \Lambda X \Lambda \), exactly 22 matrices for \( \Gamma \) leave \( X \) non-singular. For each of these values of \( \Gamma \), we find the corresponding form for \( X \) (see Appendix 1). Similarly, we substitute \( \Lambda \) and each possible \( \Gamma \) into \( \Lambda Y \Gamma = Y \) and solve the resulting system of equations. For each of the 22 remaining values of \( \Gamma \), \( Y \) must be of a certain corresponding form (see Appendix 1). Further case-by-case analysis yields exactly 16 choices of \( \Gamma \) which do not force \( (X, Y) \) to be reducible (see Appendix 1).
In each of these 16 cases, the restriction that $XYXY^2$ is diagonal forces both $X$ and $Y$ to have exactly one nonzero entry per row and column.

**Case 8:**
Suppose $\lambda_1 \neq \frac{1}{\lambda_1}$. Then $\lambda_4$, $\lambda_5$, or $\lambda_6$ must equal $\frac{1}{\lambda_1}$. We expand $\Lambda X \Lambda = X$ to get a system of equations where for any $i, j \in \{1, \ldots, 6\}$, $\lambda_i \lambda_j = 1$ if $X_{i,j} \neq 0$. If either $\lambda_1 = \frac{1}{\lambda_1}$, $\lambda_1 = \frac{1}{\lambda_5}$, or $\lambda_1 = \frac{1}{\lambda_6}$, then we find $X$ must be singular, which is a contradiction. Thus $\lambda_1 = \frac{1}{\lambda_1}$. As in Case 7, it is easy to show that there must be 0, 2, or 4 tuples with entries $\lambda_i$ where $\lambda_i = \frac{1}{\lambda_1}$ for each $i$ in that tuple. Since $\lambda_1 = \frac{1}{\lambda_1}$ and since there are at most 2 distinct solutions to $x = \frac{1}{x}$, there must be two such tuples. Without loss of generality, assume that $\lambda_4 = \frac{1}{\lambda_1}$. (Note that if instead $\lambda_5 = \frac{1}{\lambda_1}$ or $\lambda_6 = \frac{1}{\lambda_1}$, we may conjugate $X$ and $Y$ by an appropriate permutation matrix $P$.) Since $\lambda_5 \neq \frac{1}{\lambda_1}$, then $\lambda_6 = \frac{1}{\lambda_1}$. Because $\Lambda$ and $\Gamma$ are conjugate, there are $\binom{6}{3,1,1,1} = 120$ possible matrices for $\Gamma$ in terms of the $\lambda_i$’s. By case-by-case analysis, we see from $\Gamma X \Gamma = X = \Lambda X \Lambda$ that exactly 20 choices of $\Gamma$ leave $X$ non-singular. For each of the 20 possible values of $\Gamma$, we find the corresponding form of $X$ (see Appendix 1). Then we substitute $\Lambda$ and each possible $\Gamma$ into $\Lambda Y \Gamma = Y$ and solve the resulting system of equations. For each possible $\Gamma$, $Y$ must be of a certain corresponding form (see Appendix 1). Among these 20 choices of $\Gamma$, further case-by-case analysis yields exactly 6 choices of $\Gamma$ which do not force $(X,Y)$ to be reducible (see Appendix 1). In each of the 6 remaining values of $\Gamma$, we find that $XYXY^2$ is not diagonal, which is a contradiction.

**Case 9:**
Suppose that $\lambda_1 = \frac{1}{\lambda_1}$. Expand $\Lambda X \Lambda = X$ to yield a system of equations where for any $i, j \in \{1, \ldots, 6\}$, $\lambda_i \lambda_j = 1$ if $X_{i,j} \neq 0$. If $\lambda_3 \neq \frac{1}{\lambda_1}$, then either $\lambda_5 = \frac{1}{\lambda_3}$ or $\lambda_6 = \frac{1}{\lambda_3}$. In either case, $X$ must be singular, which is a contradiction. Thus $\lambda_3 = \frac{1}{\lambda_3}$. Since there are at most two distinct solutions to $x = \frac{1}{x}$, we find $\lambda_6 = \frac{1}{\lambda_3}$. There are $\binom{6}{2,2,1,1} = 180$ possible matrices for $\Gamma$ defined in terms of the $\lambda_i$’s. Substitute $\Lambda$ and each possible $\Gamma$ into $\Gamma X \Gamma = X = \Lambda X \Lambda$ and solve the resulting system of equations. By case-by-case analysis, we see that exactly 20 choices of $\Gamma$ leave $X$ non-singular. For each of these 20 possible values of $\Gamma$, we find the corresponding form of $X$ (see Appendix 1). Similarly, substitute $\Lambda$ and each possible $\Gamma$ into $\Lambda Y \Gamma = Y$ and solve the resulting system of equations. For each of the 20 remaining possible values of $\Gamma$, we find the corresponding form for $Y$ (see Appendix 1). Among these values of $\Gamma$, further case-by-case analysis yields exactly 4 choices of $\Gamma$ which do not force $(X,Y)$ to be reducible (see Appendix 1). In each of these cases, the corresponding forms of $X$ and $Y$ are $X = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 & Y_1 \\ Y_2 & 0 & 0 \\ 0 & Y_3 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & Y_1 & 0 \\ 0 & 0 & Y_2 \\ Y_3 & 0 & 0 \end{pmatrix}$, where $X_i$ and $Y_i$ are $2 \times 2$ block matrices.

Using a similar argument to that in Case 5, we reach a contradiction with both
forms of \((X, Y)\).

Hence \(\lambda_1 \neq \frac{1}{\lambda_1}\). We then see that \(\lambda_3, \lambda_5, \text{ or } \lambda_6\) equals \(\frac{1}{\lambda_1}\). Expand \(AXA = X\) to get a system of equations where for any \(i, j \in \{1, \ldots, 6\}\), \(\lambda_i \lambda_j = 1\) if \(X_{i,j} \neq 0\). If either \(\lambda_3 = \frac{1}{\lambda_1}\) or \(\lambda_6 = \frac{1}{\lambda_1}\), it follows that \(X\) is singular, which is a contradiction. Thus \(\lambda_3 = \frac{1}{\lambda_1}\). Suppose \(\lambda_5 = \frac{1}{\lambda_1}\). Then \(\lambda_6 = \frac{1}{\lambda_5}\). From \(AXA = X\), one sees that \(X_{5,5}\) and \(X_{6,6}\) are nonzero elements. Note \(\gamma_5^2 = \gamma_6^2 = 1\) since \(\Gamma XT = X\). Because \(\Lambda\) and \(\Gamma\) are conjugate, either \(\gamma_5 = \lambda_5\) and \(\gamma_6 = \lambda_6\), or \(\gamma_5 = \lambda_6\) and \(\gamma_6 = \lambda_5\). In both cases, \((X, Y)\) is not irreducible by \(\Lambda YT = Y\), which is a contradiction. Therefore \(\lambda_3 = \frac{1}{\lambda_1}\) and \(\lambda_6 = \frac{1}{\lambda_5}\). There are \(6, 2, 2, 1, 1\) = 180 possible matrices for \(\Gamma\). By case-by-case analysis, we see from \(\Gamma XT = X = \Lambda XA\) that exactly 44 choices of \(\Gamma\) leave \(X\) non-singular. For each of the 44 possible values of \(\Gamma\), we find the corresponding form of \(X\) (see Appendix 1). Substitute \(\Lambda\) and each of the possible values of \(\Gamma\) into \(\Lambda XT = Y\) and solve the resulting system of equations. For each of the 44 possible values of \(\Gamma\), we find the corresponding form of \(Y\) (see Appendix 1). Among these possible values of \(\Gamma\), further case-by-case analysis yields exactly 32 choices of \(\Gamma\) which do not force \((X, Y)\) to be reducible (see Appendix 1). For each of the 32 remaining choices of \(\Gamma\), the fact that \(XYXY^2\) is diagonal either eliminates the choice of \(\Gamma\), or forces both \(X\) and \(Y\) to have exactly one nonzero entry per row and column.

Case 10:
Assume that \(\lambda_1 \neq \frac{1}{\lambda_1}\). Then either \(\lambda_3, \lambda_4, \lambda_5, \text{ or } \lambda_6\) must equal \(\frac{1}{\lambda_1}\). We expand \(AXA = X\) to get a system of equations where for any \(i, j \in \{1, \ldots, 6\}\), \(\lambda_i \lambda_j = 1\) if \(X_{i,j} \neq 0\). If either \(\lambda_1 = \frac{1}{\lambda_3}\), \(\lambda_1 = \frac{1}{\lambda_4}\), \(\lambda_1 = \frac{1}{\lambda_5}\), \(\lambda_1 = \frac{1}{\lambda_6}\), then \(X\) must be singular, which is a contradiction. Thus \(\lambda_1 = \frac{1}{\lambda_3}\). As in cases 7 and 8, it is easy to show that there must be 1, 3, or 5 tuples with entries \(\gamma_i\) in which \(\lambda_i = \frac{1}{\lambda_1}\). Since there are at most 2 distinct solutions to \(x = \frac{1}{\lambda_1}\), there must be only one such tuple. Without loss of generality, we may assume that \(\lambda_4 = \frac{1}{\lambda_3}\) and thus \(\lambda_6 = \frac{1}{\lambda_5}\). Since \(\Lambda\) and \(\Gamma\) are conjugate, there are \(6, 2, 1, 1, 1\) = 360 possible matrices for \(\Gamma\) in terms of the \(\lambda_i\)'s. By case-by-case analysis, we see that the restrictions \(\Gamma XT = X = \Lambda XA\) and \(X\) is non-singular leave exactly 24 choices for \(\Gamma\). For each of the 24 possible values of \(\Gamma\), we find the corresponding form of \(X\) (see Appendix 1). Substitute \(\Lambda\) and each of the possible values of \(\Gamma\) into \(\Lambda YT = Y\) and solve the resulting system of equations. For each of the 24 possible values of \(\Gamma\), we find the corresponding form of \(Y\) (see Appendix 1). Among these 24 choices of \(\Gamma\), further case-by-case analysis yields exactly 8 choices of \(\Gamma\) which do not force \((X, Y)\) to be reducible (see Appendix 1). In each of these 8 cases, the restriction that \(XYXY^2\) is diagonal forces both \(X\) and \(Y\) to have exactly one nonzero entry per row and column.

Case 11:
By solving \(AXA = X\), we find that for any \(i, j \in \{1, \ldots, 6\}\), \(\lambda_i \lambda_j = 1\) if \(X_{i,j} \neq 0\). If \(X\) has more than one nonzero entry per row or column, then \(\lambda_i = \lambda_j\) for some \(i \neq j\), which is a contradiction to our assumption that all \(\lambda_i\) are distinct. Therefore since \(X\) is non-singular, \(X\) has exactly one nonzero entry per row and column. Solving
AYT = Y, we find that for any \( i, j \in \{1, \ldots, 6\} \), \( \lambda_i \gamma_j = 1 \) if \( Y_{i,j} \neq 0 \). If \( Y \) has more than one nonzero entry per row or column, then either \( \lambda_i = \lambda_j \) or \( \gamma_i = \gamma_j \) for some \( i \neq j \), both of which are contradictions. Therefore since \( Y \) is non-singular, \( Y \) has exactly one nonzero entry per row and column.

\[ \square \]

Lemma 6.2. If \((X,Y)\) satisfies the hypotheses of Lemma 3.4 where \( n = 6 \), then

\[
(X, Y) \cong \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & c_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
c_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{c_1 c_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}\]

where \( c_1, c_2 \in k \), \( c_1, c_2 \neq 0 \), and

1. \( (c_1, c_2) \neq (1,1), (-1,-1), (-1,1), (1,-1), (\zeta, \zeta), (\zeta^2, \zeta^2) \), if \( k \) does not have characteristic 2 or 3, where \( \zeta \) is a primitive cube root of unity.
2. \( (c_1, c_2) \neq (1,1), (\zeta, \zeta), (\zeta^2, \zeta^2) \), if \( k \) has characteristic 2, where \( \zeta \) is a primitive cube root of unity.
3. \( (c_1, c_2) \neq (1,1), (-1,-1), (-1,1), (1,-1) \), if \( k \) has characteristic 3.

Furthermore, for \( c_1, c_2 \) satisfying the above criteria, the following correspond to the same equivalence class: \((c_1, c_2), (\frac{1}{c_1}, \frac{1}{c_2}), (c_2, \frac{1}{c_1 c_2}), (\frac{1}{c_1 c_2}, c_1), \) and \((c_1 c_2, \frac{1}{c_1})\). If \((c'_1, c'_2)\) does not equal any of the preceding tuples, then \((c'_1, c'_2)\) represents an equivalence class distinct from \((c_1, c_2)\).

Proof. Assume \((X,Y)\) satisfies the hypotheses of Lemma 3.4. Let \( H \) be the group of \( 6 \times 6 \) matrices with exactly one nonzero entry per row and column and with the group operation of matrix multiplication. Let \( M \) be the group of all equivalence classes of \( H \) where \( A \in H \) is equivalent to \( B \in H \) when

\[
A_{i,j} = 0 \Leftrightarrow B_{i,j} = 0.
\]

Denote \([A] \in M\) as the equivalence class of \( A \in H \) and denote \([1]_M\) as the identity element in \( M \). Now consider \([X] \in M\). Since \( X^2 = I \), \([X]\) has order 1 or 2. We see that \( M \) is isomorphic to \( S_6 \). The conjugacy classes of \( S_6 \) with elements that have order 1 or 2 are represented by \((1),(12),(12)(34),(12)(34)(56)\). Thus there must be a permutation matrix, \( P \), such that \( P X P^{-1} \) is diagonal or one of

\[
X_0 = \begin{pmatrix}
0 & x_1 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & x_4 & 0 & 0 \\
0 & 0 & 0 & 0 & x_5 & 0 \\
0 & 0 & 0 & 0 & 0 & x_6
\end{pmatrix},
X_1 = \begin{pmatrix}
0 & x_1 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & x_4 & 0 & 0 \\
0 & 0 & 0 & 0 & x_5 & 0 \\
0 & 0 & 0 & 0 & 0 & x_6
\end{pmatrix},
\text{or}
\]

\[
X_2 = \begin{pmatrix}
0 & x_1 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & x_4 & 0 & 0 \\
0 & 0 & 0 & 0 & x_5 & 0 \\
0 & 0 & 0 & 0 & 0 & x_6
\end{pmatrix},
\text{for suitable choices of } x_1, \ldots, x_6.
\]
Similarly, since $Y^3 = I$, there must be a permutation matrix, $Q$, such that $QYQ^{-1}$ is diagonal or is equal to

$$Y_1 = \begin{pmatrix} 0 & y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2 & 0 & 0 & 0 \\ y_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_6 \end{pmatrix} \quad \text{or} \quad Y_2 = \begin{pmatrix} 0 & y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2 & 0 & 0 & 0 \\ y_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_6 \end{pmatrix}$$

for suitable choices of $y_1, \ldots, y_6$.

Assume $PXP^{-1}$ is diagonal. Since $P \in H$, we find $[1]_M = [PXP^{-1}] = [P][X][P^{-1}]$. Thus $[X] = [1]_M$, i.e. $X$ is a diagonal matrix. Now take an arbitrary monomial in $X$ and $Y$. Since $X^2 = I = Y^3$, this monomial can be expressed in the form $Y^{a_1}X^{y_1_a_2}X \cdots Y^{a_n}$ where $a_i = 0, 1, \text{ or } 2$ for $i = 1 \ldots n$. Because $X$ is diagonal and $Y^3 = I$,

$$[Y^{a_1}X^{y_1}a_2 \cdots Y^{a_n}] = [Y^{a_1}][X][X^{y_2}][X] \cdots [X^{y_n}] = [Y^{a_1}][1]_M[X^{y_2}][1]_M \cdots [X^{y_n}]$$

$$= [Y^{a_1}][Y^{a_2}] \cdots [Y^{a_n}] = [Y^{a_1+a_2+\ldots+a_n}] = [1]_M, [Y], \text{ or } [Y]^2.$$

Since $Y \in H$ and $Y^2 \in H$, we find that $Y$ and $Y^2$ have exactly one nonzero entry per row and column. Thus $(X, Y)$ can span at most an 18-dimensional space, which is a contradiction to $(X, Y)$ being irreducible.

Now assume $QYQ^{-1}$ is diagonal for some permutation matrix $Q$. By a similar argument to that above, we see that $Y$ is a diagonal matrix. Now take an arbitrary monomial in $X$ and $Y$. Again, this monomial can be expressed as $Y^{a_1}X^{y_1}a_2 \cdots Y^{a_n}$ where $a_i = 0, 1, \text{ or } 2$ for $i = 1 \ldots n$. Then since $Y$ is diagonal and $X^2 = I$,

$$[Y^{a_1}X^{y_1}a_2 \cdots Y^{a_n}] = [Y^{a_1}][X][X^{y_2}][X] \cdots [X^{y_n}] = [1]_M[X][1]_M[X] \cdots [1]_M$$

$$= [X][X] \cdots [X] = [X^{n-1}] = [1]_M \text{ or } [X].$$

Since $X \in H$, we find $X$ has one nonzero entry per row and column. Therefore $(X, Y)$ can span at most a 12-dimensional space, which is a contradiction to $(X, Y)$ being irreducible. Hence $PXP^{-1} = X_0, X_1$, or $X_2$ and $QYQ^{-1} = Y_1$ or $Y_2$, for some permutation matrix $Q$. We define $Y_0$ to be $PYP^{-1}$. Then for each $(X, Y)$ which satisfies the hypotheses of Lemma 3.4, we find that $(X, Y)$ must be equivalent to $(X_0, Y_0), (X_1, Y_0), \text{ or } (X_2, Y_0)$ where $Y_0$ is conjugate to $Y_1$ or $Y_2$ by a permutation matrix. Note that if $P$ is a weighted permutation matrix, then $(X, Y)$ satisfies the hypotheses of Lemma 3.4 if and only if $(PXP^{-1}, PYP^{-1})$ satisfies the hypotheses of Lemma 3.4. Thus $(X_1, Y_0)$ satisfies the hypotheses of Lemma 3.4.

After examining all possible pairs $(X_i, Y_0)$ by using the restrictions that $(X_i, Y_0)$ is irreducible and $X_i Y_0 X_i Y_0 ^2$ is diagonal, we find $(X, Y) \approx (X_2, Y_0)$, where $X_2$ is of the form above and $Y_0$ equals:

$$\begin{pmatrix} 0 & 0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_4 \\ y_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_6 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & 0 & y_4 & 0 \\ y_5 & 0 & 0 & 0 & 0 & 0 \\ y_6 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_4 & 0 \\ 0 & y_5 & 0 & 0 & 0 & 0 \\ 0 & y_6 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
for some nonzero $c$.

For suitable choices of $a$, if $X$ exists an invertible matrix and conjugation by various weighted permutation matrices, one sees that $QX = QY$.

Let $c_1, c_2, y_1, \ldots, y_6$. From computation with $X^2 = Y^3 = I$ and conjugation by various weighted permutation matrices, one sees that

$$(X, Y) \approx (X', Y') := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for some nonzero $c_1, c_2 \in k$.

We note $X^2 = I = Y^3$, $X'Y'X'Y'^2$ is diagonal, and $X'Y'^2X'Y'$ is diagonal. Thus it remains to determine which values of $c_1$ and $c_2$ lead to an irreducible $(X', Y')$ and to find the equivalence classes for $(X', Y')$. We first determine the equivalence classes for solutions $(X', Y')$.

Let $Y^* = \begin{pmatrix} c_1 & 0 \\ c_2 & 0 \\ 0 & 1 \end{pmatrix}$. Then $(X', Y^*) \approx (X', Y')$ if and only if there exists an invertible matrix $Q$ such that $QX' = X'Q$ and $QY' = Y^*Q$.

Let $Q = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & J \end{pmatrix}$ where $A$ through $J$ are $2 \times 2$ block matrices. Then by

$QX' = X'Q$, we find that $A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix}$, etc.

From $QY' = Y^*Q$, we see that:

$A = 0 \iff E = 0 \iff J = 0, \quad B = 0 \iff F = 0 \iff G = 0, \quad C = 0 \iff D = 0 \iff H = 0.$

From this result and again using $QY' = Y^*Q$, we find $(c_1', c_2') = (c_1, c_2)$ or $(\frac{1}{c_1}, \frac{1}{c_2})$ if $A \neq 0$. Likewise, $(c_1', c_2') = (c_2, \frac{1}{c_1c_2})$ or $(\frac{1}{c_2}, c_1c_2)$ if $B \neq 0$, and $(c_1', c_2') = (\frac{1}{c_2c_2}, c_1)$ or $(c_1c_2, \frac{1}{c_2})$ if $C \neq 0$. 
Note that using:

\[
Q_1 = I, \quad Q_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
Q_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & c_1c_2 \\ c_2 \end{pmatrix},
\]

\[
Q_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_6 = \begin{pmatrix} 0 & c_2 \\ c_2 & 0 \end{pmatrix},
\]

we find \(Q_iX'Q_i^{-1} = X'\) and \(Q_iY'Q_i^{-1} = Y'\) for \(i = 1 \ldots 6\), and \((c'_1, c'_2)\) as above. Since \(Q\) is invertible, \(A, B, C\) cannot all be 0. Thus \(Y'\) must have

\[ (c'_1, c'_2) = (c_1, c_2), \left(\frac{1}{c_1} \frac{1}{c_2}\right), \left(\frac{1}{c_2} \frac{1}{c_1c_2}\right), \left(\frac{1}{c_1c_2} \frac{1}{c_1}\right), \text{ or } (c_1c_2, \frac{1}{c_1}). \]

It remains to check irreducibility of \((X', Y')\).

Note that conjugating by

\[
P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}
\]

yields \((X', Y') \approx (X'', Y')\)

where

\[
X'' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}
\]

Thus \((X', Y')\) is irreducible if and only if \((X'', Y')\) is irreducible.

Let

\[
L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

and

\[
U = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

Since \((L, U)\) is irreducible, it is sufficient to generate \(L\) and \(U\) as \(k\)-linear combinations of products in \(X'', Y'\). Note that this is a necessary and sufficient condition for \((X'', Y')\) to be
irreducible.

We now find conditions that let us generate $L$ and $U$ from $X''$ and $Y'$. Let $\Lambda'' = X''Y'X''Y''$ and $\Gamma'' = X''Y'X''$. Since $Y'X'', \Lambda''Y'X'', \Lambda''\Gamma''Y'X''$, $\Gamma''Y'X'', \Lambda''Y''X''$, and $\Gamma''Y''X''$ are matrices of the form

$$
\begin{pmatrix}
0 & * & 0 & 0 & 0 \\
0 & * & 0 & 0 & 0 \\
0 & 0 & * & 0 & 0 \\
0 & 0 & 0 & * & 0 \\
* & 0 & 0 & 0 & 0
\end{pmatrix},
$$

we attempt to form $U$ and $L$ as a $k$-linear combination of those matrices. For $a_i \in k$,

$$a_1Y'X'' + a_2\Lambda''Y'X'' + a_3\Gamma''Y'X'' + a_4\Gamma''Y'X'' + a_5\Lambda''Y'X'' + a_6\Gamma''Y'X'' = U$$

if and only if $A_1$

$$\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6
\end{pmatrix} =
\begin{pmatrix}
0 \\
1 \\
1 \\
1 \\
1 \\
1
\end{pmatrix}$$

where

$$A_1 = \begin{pmatrix}
1 & c_2 & c_1c_2 & c_1c_2 & c_2^2 & c_1c_2^2 \\
c_1 & 1 & c_2^2 & c_1c_2 & c_2 & c_1c_2^2 \\
c_1 & 1 & c_2^2 & c_1 & c_2 & c_1c_2^2 \\
c_2 & c_1 & c_2^2 & c_1 & c_2 & c_1c_2^2 \\
1 & c_1 & c_2 & c_1 & c_2 & c_1c_2 \\
1 & c_1 & c_2 & c_1 & c_2 & c_1c_2^2
\end{pmatrix}.$$

Similarly, for $b_i \in k$,

$$b_1Y'X'' + b_2\Lambda''Y'X'' + b_3\Gamma''Y'X'' + b_4\Gamma''Y'X'' + b_5\Lambda''Y'X'' + b_6\Gamma''Y'X'' = L$$

if and only if $A_1$

$$\begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5 \\
b_6
\end{pmatrix} =
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.$$ Thus $U$ and $L$ can be generated as above if

$$\text{det}(A_1) = R_1/(c_1c_2^5) \neq 0,$$

where $R_1$ is a polynomial in $c_1$ and $c_2$. Likewise, we attempt to generate $U$ and $L$ using matrices $Y'X'', Y'X''\Lambda'', \Lambda''Y'X'', \Lambda''\Gamma''X'', \Lambda''\Gamma''Y'X'', \Lambda''Y''X''$, and $\Lambda''Y''X''$. Using the method above, we obtain a matrix

$$B_1 = \begin{pmatrix}
1 & \frac{1}{c_1} & c_2 & c_1c_2 & c_1c_2^2 & c_2^2 \\
c_1 & \frac{1}{c_1} & 1 & c_1c_2 & c_2 & c_1c_2^2 \\
c_1 & \frac{1}{c_1} & c_2 & c_1c_2 & c_2 & c_1c_2^2 \\
c_2 & 1 & c_1c_2 & c_2 & c_1c_2 & c_2 \\
1 & c_1c_2 & c_1 & c_2 & c_1 & c_1c_2 \\
1 & c_1 & c_2 & c_1 & c_2 & c_1c_2
\end{pmatrix}.$$

We are able to generate $L$ and $U$ from the monomials above if $\text{det}(B_1) = R_2/(c_1c_2^5) \neq 0$, where $R_2$ is a polynomial in $c_1$ and $c_2$. Thus, if we are not able to generate $L$
and $U$ in either way, $R_1 = R_2 = 0$.

Earlier, we concluded that $(X', Y')$ is irreducible if and only if $(X'', Y')$ is irreducible. Also, $(X'', Y')$ is irreducible if and only if $L$ and $U$ can be generated from $X''$ and $Y'$. From this, we are able to conclude that if $(X', Y')$ is not irreducible, then $R_1 = R_2 = 0$.

We now solve $R_1 = R_2 = 0$. By factoring $R_2$ we find

$$R_2 = (1 - c_1)(c_2 - 1)(c_1c_2 - 1)F_1F_2F_3$$

where $F_1$, $F_2$, $F_3$ are irreducible polynomials in $c_1$, $c_2$. Our explicit choices of $F_1$, $F_2$, and $F_3$ can be found in Appendix 2. Then $R_1 = R_2 = 0$ if and only if $R_1 = 0$ and at least one of the following conditions hold: $c_1 = 1$, $c_2 = 1$, $c_1c_2 = 1$, $F_1 = 0$, $F_2 = 0$, or $F_3 = 0$.

- If $R_1 = 0$ and $c_1 = 1$ then $(c_2 - 1)^6(c_2^2 - 1)^3 = 0$.
- If $R_1 = 0$ and $c_2 = 1$ then $(c_1 - 1)^6(c_1^2 - 1)^3 = 0$.
- If $R_1 = 0$ and $c_1c_2 = 1$ then $(c_2 - 1)^6(c_2^2 - 1)^3 = 0$.

Thus if $R_1 = 0$ and either $c_1 = 1$, $c_2 = 1$, or $c_1c_2 = 1$, then $(c_1, c_2) = (1, 1), (1, -1), (-1, 1), or (-1, -1) in fields not of characteristic 2 and $(c_1, c_2) = (1, 1)$ in fields of characteristic 2.

We next repeatedly use the command sprem in Maple to find a univariate polynomial in $c_1$ or $c_2$ which is contained in the ideal $\langle R_1, F_i \rangle$ of $k[c_1, c_2]$ for each $i = 1, 2, 3$. See Appendix 2 for the detailed Maple commands used in this section of the paper. Specifically, sprem inputs a variable $x$, and multivariate polynomials in $x$, say $a$ and $b$. It computes multivariate polynomials in $x$ with integer coefficients, say $m$ and $q$, where $ma = bq + r$, and the degree of $x$ in $r$ is strictly less than the degree of $x$ in $b$.

The output of sprem is the multivariate function $r$. $m$ is always of the form $x^n$ for some $n$. Temporarily regard $c_1$ and $c_2$ as indeterminates, and temporarily replace $R_1$ and $F_1$ with their natural preimages in $Z[c_1, c_2]$. Using the function sprem to recursively reduce the degree of $c_2$ when starting with initial polynomials $R_1$ and $F_1$, we see that $c_1^{24}(c_1 - 1)^{24}(c_1^2 + c_1 + 1)^6$ is in the ideal $\langle R_1, F_1 \rangle$ of $Z[c_1, c_2]$. Now let $c_1$, $c_2$, $R_1$, and $F_1$ again be elements of $k$. It follows that $c_1^{24}(c_1 - 1)^{24}(c_1^2 + c_1 + 1)^6$ is in the ideal $\langle R_1, F_1 \rangle$ of $k[c_1, c_2]$. Hence if $R_1 = F_1 = 0$, either $c_1 = 0$, $c_1 = 1$, or $c_1^2 + c_1 + 1 = 0$.

Similarly, we temporarily regard $c_1$ and $c_2$ as indeterminates, and temporarily replace $R_1$ and $F_2$ with their natural preimages in $Z[c_1, c_2]$. We use the Maple command sprem to recursively reduce the degree of $c_2$ when starting with initial polynomials $R_1$ and $F_2$. Our computation yield that $(c_1 - 1)^{24}(c_1^2 + c_1 + 1)^6(c_1^2 - c_1 + 1)^{11}$ is in the ideal $\langle R_1, F_2 \rangle$ of $Z[c_1, c_2]$. Returning $c_1$, $c_2$, $R_1$, and $F_2$ to their original form, we conclude that $(c_1 - 1)^{24}(c_1^2 + c_1 + 1)^6(c_1^2 - c_1 + 1)^{11}$ is in the ideal $\langle R_1, F_2 \rangle$ of $k[c_1, c_2]$. Hence if $R_1 = F_2 = 0$, either $c_1 = 1$, $c_1^2 + c_1 + 1 = 0$, or $c_1^2 - c_1 + 1 = 0$.

Again, temporarily regard $c_1$ and $c_2$ as indeterminates, and temporarily replace $R_1$ and $F_3$ with their natural preimages in $Z[c_1, c_2]$. We now solve $R_1 = F_3 = 0$ in two different ways. First, we recursively reduce the degree of $c_2$ by using the Maple command sprem with initial polynomials $R_1$ and $F_3$. We find

$$c_1^{186}(c_1 + 1)^{36}(c_1 - 1)^{56}(c_1^2 + c_1 + 1)^{57}T_1^4T_2^2$$

is in the ideal $\langle R_1, F_3 \rangle \subset Z[c_1, c_2]$ where $T_1$ is an irreducible polynomial in $c_1$ of degree 28 and where $T_2$ is an irreducible polynomial in $c_1$ of degree 40. Alternatively, factor $R_1$ into $(c_2 - c_1)(c_2^2 - 1)(c_1c_2^2 - 1)R_3$, where $R_3$ is a nonhomogeneous polynomial in $c_1$ and $c_2$ of total degree 14. Now we use the Maple command sprem
to recursively reduce the degree of $c_2$ using initial polynomials $F_3$ and each of the polynomials $(c_2 - c_1)$, $(c_2^2c_2 - 1)$, $(c_1^2 - 1)$, and $R_3$. In respective order, we find that the following polynomials are in $(R_1, F_3) \subset \mathbb{Z}[c_1, c_2]$:

- $(c_1 + 1)^2(c_1 - 1)^2(c_1^2 + c_1 + 1)^2$
- $c_1^2(c_1 + 1)^2(c_1 - 1)^2(c_1^2 + c_1 + 1)^2$
- $c_1^2(c_1^2 + c_1 + 1)^2$
- $c_1^{10}(c_1 + 1)^{16}(c_1 - 1)^{44}(c_1^2 + c_1 + 1)^{31}T_3^4T_4^2$

where $T_3$ is an irreducible polynomial of degree 16 and $T_4$ is an irreducible polynomial of degree 26. Now remove the temporary replacements of $c_1$, $c_2$, $R_1$, and $F_3$. By comparing solutions obtained in the two different ways, we conclude that if $R_1 = F_3 = 0$, then $c_1 = 0$, $c_1 - 1 = 0$, $c_1 + 1 = 0$, or $c_1^2 + c_1 + 1 = 0$.

Thus if $R_1 = 0$ and either $F_1$, $F_2$, or $F_3 = 0$, then $c_1 = 0$, $c_1 = 1$, $c_1 + 1 = 0$, $c_1^2 + c_1 + 1 = 0$, or $c_1^2 - c_1 + 1 = 0$. We reject the case that $c_1 = 0$ since this forces $Y^3 \neq I$. The case where $c_1 = 1$ was solved above. If $k$ is not of characteristic 2, solving $c_1 + 1 = 0$ gives $c_1 = -1$. If $c_1 = -1$ and $R_1 = R_2 = 0$, then $c_1 = 1$ or $c_2 = -1$. These solutions are both listed above. If $k$ is of characteristic 2 and $c_1 + 1 = 0$ or $k$ is of characteristic 3 and $c_1^2 + c_1 + 1 = 0$ then $c_1 = 1$, which was solved above. Also, if $k$ is of characteristic 3 and $c_1^2 - c_1 + 1 = 0$ then $c_1 = -1$, which was solved above. It remains to consider fields not of characteristic 3 where $c_1^2 + c_1 + 1 = 0$ or $c_1^2 - c_1 + 1 = 0$.

Thus assume $k$ is not of characteristic 3, $c_1^2 + c_1 + 1 = 0$, and $R_1 = R_2 = 0$. Temporarily replace $c_1$, $c_2$, $R_1$, and $R_2$ with their natural preimages in $\mathbb{Z}[c_1, c_2]$. We use the Maple command sprem to recursively reduce the degree of $c_1$ with initial polynomials $R_1$ and $c_1^2 + c_1 + 1$, and $R_2$ and $c_1^2 + c_1 + 1$. We find that:

\[
(c_1^2 - c_2 + 1)(c_2^6 - 5c_2^5 + 23c_2^4 - 8c_2^3 - c_2^2 - 2c_2 + 1)(c_1^2 + c_2 + 1)^5 \\
(c_2^6 - 2c_2^5 - c_2^4 - 8c_2^3 + 23c_2^2 - 5c_2 + 1) \in (R_1, c_1^4 + c_1 + 1), \text{ and} \\
27c_2^2(c_2^2 - 2c_2 + 4)(4c_2^2 - 2c_2 + 1)(c_1^2 + c_2 + 1)^2(c_2^2 + c_2 + 1)^5 \in (R_2, c_1^2 + c_1 + 1).
\]

Note that if $R_1 = R_2 = 0$ and $c_1^2 + c_1 + 1 = 0$, then $c_2^2 + c_2 + 1 = 0$. Now remove the temporary replacements of $c_1$, $c_2$, $R_1$, and $R_2$. Thus $c_1 = \zeta$ or $\zeta^2$ and $c_2 = \zeta$ or $\zeta^2$, where $\zeta^3 = 1$. Since $R_1 \neq 0$ when $(c_1, c_2) = (\zeta, \zeta^2)$, $(\zeta^2, \zeta)$, we find that $(c_1, c_2) = (\zeta, \zeta)$ or $(\zeta^2, \zeta^2)$.

Now assume $k$ is not of characteristic 3, $c_1^2 - c_1 + 1 = 0$, and $R_1 = R_2 = 0$. Temporarily regard $c_1$ and $c_2$ as indeterminates, and temporarily replace $R_1$ and $R_2$ with their natural preimages in $\mathbb{Z}[c_1, c_2]$. Using the Maple command sprem, we recursively reduce the degree of $c_1$ with initial polynomials $R_1$ and $c_1^2 - c_1 + 1$, and $R_2$ and $c_1^2 - c_1 + 1$. As a result:

\[
(c_1^2 - c_2 + 1)(c_2^2 + c_2 + 1)(c_2^4 - c_2 + 1)(c_1^2 - 3c_1^2 + 9c_1^2 + 4c_1^2 - 18c_1^2 + 15c_1^2 - 6c_2 + 1) \\
(c_2^6 - 6c_2^5 + 15c_2^4 - 18c_2^3 + 4c_2^4 + 9c_2^3 - 3c_2 + 1) \in (R_1, c_1^2 - c_1 + 1), \text{ and} \\
c_2^2(3c_2^2 - 3c_2 + 1)(c_2^2 - 3c_2 + 3)(c_2^2 - c_2 + 1)(c_2 - 1)^2(4c_2^2 + 6c_2^2 + c_2^2 - 3c_2 + 1) \\
(c_2^4 - 3c_2^3 + c_2^2 + 6c_2 + 4) \in (R_2, c_1^2 - c_1 + 1).
\]
Note that if \( R_1 = R_2 = 0 \) and \( c_1^2 - c_1 + 1 = 0 \) then \( c_2^2 - c_2 + 1 = 0 \). Now remove the temporary replacements of \( c_1, c_2, R_1, \) and \( R_2 \). Thus if \( R_1 = R_2 = 0 \), then \( c_1 = -\zeta \) or \(-\zeta^2\) and \( c_2 = -\zeta \) or \(-\zeta^2\), where \( \zeta^3 = 1 \). However, if \( (c_1, c_2) = (-\zeta, -\zeta) \) or \( (-\zeta^2, -\zeta^2) \), then \( R_2 \neq 0 \), which is a contradiction. Also, if \( (c_1, c_2) = (-\zeta^2, -\zeta) \) or \( (-\zeta, -\zeta) \), then \( R_1 \neq 0 \), which is a contradiction.

We now consider all solutions to \( R_1 = R_2 = 0 \).

- \( (c_1, c_2) = (1, 1), (\zeta, \zeta), (\zeta^2, \zeta^2) \) if \( k \) is of characteristic 2,
- \( (c_1, c_2) = (1, 1), (1, -1), (-1, 1), (-1, -1) \), if \( k \) is of characteristic 3,
- \( (c_1, c_2) = (1, 1), (1, -1), (-1, -1), \zeta, \zeta, (\zeta^2, \zeta^2) \) if \( k \) is not of characteristic 2 or 3.

Thus if \((X', Y')\) is not irreducible, \((c_1, c_2)\) is among the preceding pairs. The pair of matrices \((X', Y')\) given when \((c_1, c_2) = (-1, -1), (1, 1)\), and \((1, -1)\) are all equivalent. Also, the pair of matrices \((X', Y')\) when \((c_1, c_2) = (\zeta, \zeta)\) and \((\zeta^2, \zeta^2)\) are equivalent. Thus it is sufficient to check irreducibility of \((X', Y')\) when \((c_1, c_2) = (1, 1), (1, -1), \zeta, \zeta, \zeta, \zeta, (\zeta^2, \zeta^2)\).

If \( c_1 = c_2 = 1 \) then \((X'Y')^2 = (Y'X')^2\). Let \( A = X'Y' \) and \( B = Y'X' \). Then \((X', Y')\) is irreducible if and only if \((A, B)\) is irreducible. Note \( AB = AB \) yields \( A^6 = B^6 = I \) by Remark 3.3. Also, \( ABA = BAB \). From these relations, we can count possible monomials in \( A \) and \( B \) to find that \((A, B)\) span at most an 18-dimensional space. Thus \((A, B)\) is not irreducible, and hence \((X', Y')\) is not irreducible.

If \( c_1 = c_2 = -1 \) and 

\[
Q = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0
\end{pmatrix},
\]

then

\[
(QX'Q^{-1}, QY'Q^{-1}) = \begin{pmatrix}
1 & \quad & \quad & \quad & \quad & \quad \\
1 & \quad & \quad & \quad & \quad & \quad \\
1 & \quad & \quad & \quad & \quad & \quad \\
-1 & \quad & \quad & \quad & \quad & \quad \\
-1 & \quad & \quad & \quad & \quad & \quad \\
-1 & \quad & \quad & \quad & \quad & \quad
\end{pmatrix},
\]

\[\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Since \((QX'Q^{-1}, QY'Q^{-1})\) is not irreducible, neither is \((X', Y')\) when \( c_1 = c_2 = -1 \). This gives the desired result.

\textbf{Remark 6.3.} In Lemma 6.2, we proved further that if

\[
(X, Y) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

where

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & c_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
c_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]
(1) \((c_1, c_2) \neq (1, 1), (-1, -1), (-1, 1), (1, -1), (\zeta, \zeta^2), (\zeta^2, \zeta^2)\), if \(k\) does not have characteristic 2 or 3, where \(\zeta\) is a primitive cube root of unity;

(2) \((c_1, c_2) \neq (1, 1), (\zeta, \zeta), (\zeta^2, \zeta^2)\), if \(k\) has characteristic 2, where \(\zeta\) is a primitive cube root of unity;

(3) \((c_1, c_2) \neq (1, 1), (-1, -1), (-1, 1), (1, -1)\), if \(k\) has characteristic 3, then \((X, Y)\) satisfies the hypotheses of Lemma 3.4.

6.4 If \(\rho : PSL_2(\mathbb{Z}) \to GL_n(k)\) is an irreducible 6-dimensional representation of \(PSL_2(\mathbb{Z}) = \langle x, y \mid x^2 = y^3 = 1 \rangle\) with \(X := \rho(x)\) and \(Y := \rho(y)\), it follows from Lemma 6.1 that \(X\) and \(Y\) have one nonzero entry per row and column. In particular, \(\rho : PSL_2(\mathbb{Z}) \to GL_n(k)\) is an irreducible 6-dimensional representation of \(PSL_2(\mathbb{Z})\) if and only if \((X, Y)\) satisfies the hypothesis of Lemma 3.4. Then by Lemma 6.2 and Remark 6.3, we have classified the irreducible 6-dimensional representations of \(PSL_2(\mathbb{Z})\) up to equivalence.

7. Appendix 1

In this Appendix, we list the matrices used in the case-by-case analysis proofs in Lemma 6.1. Again, let \(\gamma_i\) denote \(\Gamma_{i,i}\), and \(\lambda_i\) denote \(\Lambda_{i,i}\). Let \((a_1, a_2, a_3, a_4, a_5, a_6)\) denote the unique \(6 \times 6\) permutation matrix which maps \(e_i\) to \(e_{a_i}\), where \(e_i\) is the standard basis vector of \(k^6\).

Case 5: In our proof, we found that \(\lambda_1 = \frac{1}{\lambda^5}\) and \(\lambda_5 = \frac{1}{\lambda^6}\). From \(\Gamma X \Gamma = X\), we conclude that if \(\gamma_i \neq 1\), then \(X_{i,j} = 0\). From \(AXA = X\), we conclude that if \(\lambda_i \lambda_j \neq 1\), then \(X_{i,j} = 0\). For each of the 30 matrices for \(\Gamma\) in terms of \(\lambda_i\), we determine which entries of \(X\) must equal 0. Of the 30 possible matrices for \(\Gamma\), we found that for all but 14 matrices, \(X\) is forced to be singular. Two of these \(\Gamma\) lead to \((X, Y)\) irreducible. For the 12 remaining matrices for \(\Gamma\), we replace \((X, Y)\) with \((PXP^{-1}, PYP^{-1})\) for an appropriate permutation matrix \(P\), so that \(\Lambda\) is preserved, \(\gamma_1 = \gamma_2 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_3 = \lambda_5, \gamma_4 = \lambda_6\). We list each of the 14 possible matrices for \(\Gamma\), and for the 12 possible matrices for \(\Gamma\) which do not lead to \((X, Y)\) irreducible, we list the corresponding permutation matrix \(P\).

(1) \(\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \lambda_1, \gamma_5 = \lambda_5, \gamma_6 = \lambda_6\); \(\Lambda = \Gamma\) implies \((X, Y)\) reducible.

(2) \(\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \lambda_1, \gamma_5 = \lambda_6, \gamma_6 = \lambda_5\); \(\Lambda \Gamma = I\) implies \((X, Y)\) reducible.

(3) \(\gamma_1 = \gamma_2 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_3 = \lambda_5, \gamma_4 = \lambda_6\); \(P\) is the identity matrix.

(4) \(\gamma_1 = \gamma_2 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_3 = \lambda_6, \gamma_4 = \lambda_5, \gamma_6 = \lambda_5; P = (1, 2, 4, 3, 5, 6)\).

(5) \(\gamma_1 = \gamma_3 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_2 = \lambda_5, \gamma_4 = \lambda_6, \gamma_6 = \lambda_5; P = (1, 2, 4, 3, 5, 6)\).

(6) \(\gamma_1 = \gamma_4 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_2 = \lambda_5, \gamma_4 = \lambda_6, \gamma_6 = \lambda_5; P = (1, 2, 4, 3, 5, 6)\).

(7) \(\gamma_1 = \gamma_4 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_2 = \lambda_5, \gamma_4 = \lambda_6, \gamma_6 = \lambda_5; P = (1, 2, 4, 3, 5, 6)\).

(8) \(\gamma_1 = \gamma_3 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_2 = \lambda_6, \gamma_4 = \lambda_5, \gamma_6 = \lambda_5; P = (1, 3, 4, 2, 5, 6)\).

(9) \(\gamma_2 = \gamma_3 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_1 = \lambda_5, \gamma_4 = \lambda_6, \gamma_6 = \lambda_5; P = (2, 3, 4, 1, 5, 6)\).

(10) \(\gamma_2 = \gamma_4 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_1 = \lambda_5, \gamma_3 = \lambda_6, \gamma_6 = \lambda_5; P = (2, 4, 1, 3, 5, 6)\).

(11) \(\gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_1 = \lambda_5, \gamma_2 = \lambda_6, \gamma_6 = \lambda_5; P = (3, 4, 1, 2, 5, 6)\).

(12) \(\gamma_3 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_1 = \lambda_6, \gamma_3 = \lambda_5, \gamma_6 = \lambda_5; P = (2, 3, 4, 1, 5, 6)\).

(13) \(\gamma_2 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_1 = \lambda_6, \gamma_3 = \lambda_5, \gamma_6 = \lambda_5; P = (2, 4, 3, 1, 5, 6)\).

(14) \(\gamma_3 = \gamma_5 = \gamma_6 = \lambda_1, \gamma_1 = \lambda_6, \gamma_2 = \lambda_5, \gamma_6 = \lambda_5; P = (3, 4, 2, 1, 5, 6)\).

Case 7: Here we list the 22 possible values for \(\Gamma\) and the corresponding forms of \(X\) and \(Y\). The forms of \(X\) and \(Y\) are described by which entries (of \(X\) or \(Y\) respectively) must equal 0. There are 5 forms for \(X\) and 22 forms for \(Y\).
\[(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) =
\begin{align*}
(1) \quad & (\lambda_1, \lambda_3, \lambda_5, \lambda_1, \lambda_3, \lambda_5) \\
(2) \quad & (\lambda_1, \lambda_5, \lambda_3, \lambda_1, \lambda_3, \lambda_5) \\
(3) \quad & (\lambda_1, \lambda_3, \lambda_5, \lambda_1, \lambda_3, \lambda_5) \\
(4) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(5) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(6) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(7) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(8) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(9) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(10) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(11) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(12) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(13) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(14) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(15) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(16) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(17) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(18) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(19) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(20) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(21) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1) \\
(22) \quad & (\lambda_3, \lambda_5, \lambda_3, \lambda_1, \lambda_5, \lambda_1)
\end{align*}
\]

For \(\Gamma\) in cases 1 and 6, \(X\) has the following form: if \(X_{i,j} \neq 0\), then \((i, j) = (1, 1), (1, 2), (2, 1), (2, 2), (3, 5), (3, 6), (4, 5), (4, 6), (5, 3), (5, 4), (6, 3),\) or \((6, 4)\).

For \(\Gamma\) in cases 2 and 4, \(X\) has the following form: if \(X_{i,j} \neq 0\), then \((i, j) = (1, 1), (1, 2), (2, 1), (2, 2), (3, 5), (4, 6), (5, 3),\) or \((6, 4)\).

For \(\Gamma\) in cases 3 and 5, \(X\) has the following form: if \(X_{i,j} \neq 0\), then \((i, j) = (1, 1), (1, 2), (2, 1), (2, 2), (3, 6), (4, 5), (5, 4),\) or \((6, 3)\).

For \(\Gamma\) in cases 7, 9, 12, 14, 15, 17, 20, and 22, \(X\) has the following form: if \(X_{i,j} \neq 0\), then \((i, j) = (1, 2), (2, 1), (3, 5), (4, 6), (5, 3),\) or \((6, 4)\).

For \(\Gamma\) in cases 8, 10, 11, 13, 16, 18, 19, and 21, \(X\) has the following form: if \(X_{i,j} \neq 0\), then \((i, j) = (1, 2), (2, 1), (3, 6), (4, 5), (5, 4),\) or \((6, 3)\).

Fix \(\Gamma\) in one of the above cases. Then \(Y\) has the following form: if \(Y_{i,j} \neq 0\), then \(\lambda_i \gamma_j = 1\). Note that \(\lambda_1 = \frac{1}{\lambda_3}\) and \(\lambda_3 = \frac{1}{\lambda_2}\).

The selection of \(\Gamma\) in the cases 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, and 22 does not force \((X, Y)\) to be reducible.

**Case 8:** Below are the 20 possible values for \(\Gamma\) and the corresponding forms of \(X\) and \(Y\). The forms of \(X\) and \(Y\) are described by which entries (of \(X\) or \(Y\) respectively) must equal 0. There are 7 forms for \(X\) and 20 forms for \(Y\).

\[(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) =
\begin{align*}
(1) \quad & (\lambda_1, \lambda_5, \lambda_6, \lambda_2, \lambda_1, \lambda_1) \\
(2) \quad & (\lambda_1, \lambda_6, \lambda_5, \lambda_1, \lambda_1, \lambda_1) \\
(3) \quad & (\lambda_5, \lambda_1, \lambda_6, \lambda_4, \lambda_1, \lambda_1) \\
(4) \quad & (\lambda_5, \lambda_6, \lambda_1, \lambda_2, \lambda_1, \lambda_1) \\
(5) \quad & (\lambda_6, \lambda_1, \lambda_5, \lambda_1, \lambda_1, \lambda_1) \\
(6) \quad & (\lambda_6, \lambda_5, \lambda_1, \lambda_4, \lambda_1, \lambda_1) \\
(7) \quad & (\lambda_2, \lambda_5, \lambda_6, \lambda_1, \lambda_1, \lambda_1) \\
(8) \quad & (\lambda_2, \lambda_6, \lambda_5, \lambda_1, \lambda_1, \lambda_1) \\
(9) \quad & (\lambda_5, \lambda_4, \lambda_6, \lambda_1, \lambda_1, \lambda_1) \\
(10) \quad & (\lambda_5, \lambda_6, \lambda_4, \lambda_1, \lambda_1, \lambda_1) \\
(11) \quad & (\lambda_6, \lambda_4, \lambda_5, \lambda_1, \lambda_1, \lambda_1) \\
(12) \quad & (\lambda_6, \lambda_5, \lambda_4, \lambda_1, \lambda_1, \lambda_1) \\
(13) \quad & (\lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1) \\
(14) \quad & (\lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1) \\
(15) \quad & (\lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1) \\
(16) \quad & (\lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1) \\
(17) \quad & (\lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1) \\
(18) \quad & (\lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1) \\
(19) \quad & (\lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1) \\
(20) \quad & (\lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1)
\end{align*}
\]
Fix \( \Gamma \) in one of the above cases. Then \( Y \) has the following form: if \( Y_{i,j} \neq 0 \), then \( \lambda_i \gamma_j = 1 \). Note that \( \lambda_1 = \frac{1}{\lambda_1} \), \( \lambda_4 = \frac{1}{\lambda_4} \), and \( \lambda_5 = \frac{1}{\lambda_5} \).

The selection of \( \Gamma \) in the cases 7, 8, 9, 10, 11, and 12, does not force \((X, Y)\) to be reducible.

**Case 9:** Here, we list the possible values for \( \Gamma \) and the corresponding forms of \( X \) and \( Y \) for two values of \( \Lambda \).

First we assume \( \lambda_1 = \frac{1}{\lambda_1} \), \( \lambda_3 = \frac{1}{\lambda_3} \), and \( \lambda_5 = \frac{1}{\lambda_5} \). There are 20 possible values of \( \Gamma \). The forms of \( X \) and \( Y \) are described by which entries (of \( X \) or \( Y \) respectively) must equal 0. There are 4 forms for \( X \) and 20 forms for \( Y \).

\[
(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) =
\begin{align*}
(1) & (\lambda_3, \lambda_3, \lambda_5, \lambda_6, \lambda_1, \lambda_1) \\
(4) & (\lambda_6, \lambda_5, \lambda_3, \lambda_3, \lambda_1, \lambda_1) \\
(7) & (\lambda_5, \lambda_6, \lambda_1, \lambda_1, \lambda_3, \lambda_3) \\
(10) & (\lambda_1, \lambda_1, \lambda_3, \lambda_6, \lambda_5) \\
(13) & (\lambda_1, \lambda_3, \lambda_3, \lambda_1, \lambda_5, \lambda_6) \\
(16) & (\lambda_3, \lambda_1, \lambda_1, \lambda_3, \lambda_6, \lambda_5) \\
(19) & (\lambda_3, \lambda_3, \lambda_1, \lambda_1, \lambda_5, \lambda_6)
\end{align*}
\] (1)

For \( i, j \) = \((1,2), (2,1), (3,3), (3,4), (4,3), (4,4), (5,6), \) or \((6,5)\).

For \( \Gamma \) in cases 1, 2, 5, and 6, \( X \) has the following form: if \( X_{i,j} \neq 0 \), then \((i, j) = (1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (5, 6), \) or \((6, 5)\).

For \( \Gamma \) in cases 9, 10, 19, and 20, \( X \) has the following form: if \( X_{i,j} \neq 0 \), then \((i, j) = (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 6), \) or \((6, 5)\).

For \( \Gamma \) in cases 11, 12, 13, 14, 15, 16, 17, and 18, \( X \) has the following form: if \( X_{i,j} \neq 0 \), then \((i, j) = (1, 1), (2, 2), (3, 3), (4, 4), (5, 6), \) or \((6, 5)\).

Fix \( \Gamma \) in one of the above cases. Then \( Y \) has the following form: if \( Y_{i,j} \neq 0 \), then \( \lambda_i \gamma_j = 1 \). Note that the selection of \( \Gamma \) in the cases 1, 2, 7, and 8, does not force \((X, Y)\) to be reducible.

Now assume \( \lambda_1 = \frac{1}{\lambda_1} \) and \( \lambda_5 = \frac{1}{\lambda_5} \). There are 44 possible values of \( \Gamma \). The forms of \( X \) and \( Y \) are described by which entries (of \( X \) or \( Y \) respectively) must equal 0. There are 3 forms for \( X \) and 44 forms for \( Y \).

\[
(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) =
\]
For $\Gamma$ in cases 1, 3, 5, 7, 10, 12, 14, 16, 17, 19, 21, 23, 26, 28, 30, 32, 35, 36, 41, and 42, $X$ has the following form: if $X_{i,j} \neq 0$, then $(i,j) = (1,3), (2,4), (3,1), (4,2), (5,6)$, or $(6,5)$.

For $\Gamma$ in cases 2, 4, 6, 8, 9, 11, 13, 15, 18, 20, 22, 24, 25, 27, 29, 31, 34, 37, 40, and 43 $X$ has the following form: if $X_{i,j} \neq 0$, then $(i,j) = (1,4), (2,3), (3,2), (4,1), (5,6)$, or $(6,5)$.

For $\Gamma$ in cases 33, 38, 39, and 44, $X$ has the following form: if $X_{i,j} \neq 0$, then $(i,j) = (1,3), (1,4), (2,3), (2,4), (3,1), (3,2), (4,1), (4,2), (5,6)$, or $(6,5)$. Fix $\Gamma$ in one of the above cases. Then $Y$ has the following form: if $Y_{i,j} \neq 0$, then $\lambda_{i} \gamma_{j} = 1$. We note that the selection of $\Gamma$ in the cases 1 through 32 does not force $(X,Y)$ to be reducible.

**Case 10:** Listed below are the 24 possible values for $\Gamma$ and the corresponding forms of $X$ and $Y$. The forms of $X$ and $Y$ are described by which entries (of $X$ or $Y$ respectively) must equal 0. There are 2 forms for $X$ and 24 forms for $Y$.

\[
\begin{pmatrix}
\gamma_{1}, \gamma_{2}, \gamma_{3}, \gamma_{4}, \gamma_{5}, \gamma_{6}
\end{pmatrix} =
\begin{pmatrix}
1, 1, 1, 1, 1, 1
\end{pmatrix}
\]

For $\Gamma$ in cases 3, 4, 7, 8, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, and 24, $X$ has the following form: if $X_{i,j} \neq 0$, then $(i,j) = (1,2), (2,1), (3,4), (4,3), (5,6)$, or $(6,5)$.

For $\Gamma$ in cases 1, 2, 5, 6, 9, 10, 13, and 14, $X$ has the following form: if $X_{i,j} \neq 0$, then $(i,j) = (1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (5,6)$, or $(6,5)$.

Fix $\Gamma$ in one of the above cases. Then $Y$ has the following form: if $Y_{i,j} \neq 0$, then $\lambda_{i} \gamma_{j} = 1$. Note that $\lambda_{1} = \frac{1}{\lambda_{2}}$, $\lambda_{3} = \frac{1}{\lambda_{4}}$, and $\lambda_{5} = \frac{1}{\lambda_{6}}$. Also, the selection of $\Gamma$ in the cases 3, 4, 7, 8, 17, 18, 19, and 20, does not force $(X,Y)$ to be reducible.
In this Appendix, we list the specific choices of polynomials and the Maple commands used in the latter half of the proof of Lemma 6.2.

Our choices of \( F_1, F_2, \) and \( F_3 \) are:

\[
F_1 = c_1^2 c_2^2 - c_1 c_2 - c_1 - c_2 + 1
\]
\[
F_2 = c_1^2 c_2 - c_1^2 + c_1 c_2 - c_1 c_2^2 + c_1 c_2 + c_1^2 - c_1^2 c_2^2 - c_1^2 c_2^4 - 6c_1^2 c_2^2
\]
\[
F_3 = c_1^4 c_2^4 + c_1^4 + c_1 c_2^2 + c_1^2 c_2^2 - c_1^2 c_2^4 + c_1^2 c_2^2 + c_1^2 c_2^4 - c_1^2 c_2^4 - 6c_1^2 c_2^2
\]

For multivariate polynomials \( A \) and \( B \) in the indeterminate \( x \), \text{sprem}(A, B, x, \text{‘m’, ‘q’}) outputs a multivariate polynomial \( r \), where \( mA = qB + r \) and \( m, q, r \) are polynomials over \( \mathbb{Z} \). The variables \( m \) and \( q \) are assigned their corresponding polynomial values. Furthermore, the degree of \( x \) in \( r \) is strictly less than the degree of \( x \) in \( B \). We use the command \text{‘factor’} to determine the roots of the polynomials in question.

To solve \( R_1 = F_1 = 0 \):

\[
\begin{align*}
A1 & := \text{sprem}(R1, F1, c2, \text{‘m’, ‘q’}); \\
\text{sprem}(F1, A1, c2, \text{‘m’, ‘q’}); \\
\text{factor}(\%);
\end{align*}
\]

To solve \( R_1 = F_2 = 0 \):

\[
\begin{align*}
A2 & := \text{sprem}(R1, F2, c2, \text{‘m’, ‘q’}); \\
\text{sprem}(F2, A2, c2, \text{‘m’, ‘q’}); \\
\text{factor}(\%);
\end{align*}
\]

To solve \( R_1 = F_3 = 0 \):

Without factoring \( R_1 \):

\[
\begin{align*}
A3 & := \text{sprem}(R1, F3, c2, \text{‘m’, ‘q’}); \\
A4 & := \text{sprem}(F3, A3, c2, \text{‘m’, ‘q’}); \\
A5 & := \text{sprem}(A3, A4, c2, \text{‘m’, ‘q’}); \\
\text{sprem}(A4, A5, c2, \text{‘m’, ‘q’}); \\
\text{factor}(\%);
\end{align*}
\]

By factoring \( R_1 \):

\[
\begin{align*}
\text{factor}(R1); \\
\text{sprem}(F3, c1^2 * c2 - 1, c2, \text{‘m’, ‘q’}); \\
\text{factor}(\%); \\
\text{sprem}(F3, c2 - c1, c2, \text{‘m’, ‘q’}); \\
\text{factor}(\%); \\
A6 & := \text{sprem}(F3, c1 * c2^2 - 1, c2, \text{‘m’, ‘q’}); \\
\text{sprem}(c1 * c2^2 - 1, A6, c2, \text{‘m’, ‘q’}); \\
\text{factor}(\%); \\
A7 & := \text{sprem}(R3, F3, c2, \text{‘m’, ‘q’}); \\
A8 & := \text{sprem}(F3, A7, c2, \text{‘m’, ‘q’}); \\
A9 & := \text{sprem}(A7, A8, c2, \text{‘m’, ‘q’}); \\
\text{sprem}(A8, A9, c2, \text{‘m’, ‘q’}); \\
\text{factor}(\%);
\end{align*}
\]
To solve $R_1 = R_2 = c_1^2 + c_1 + 1 = 0$:

We first solve $R_1 = c_1^2 + c_1 + 1 = 0$.

> B1 := sprem(R1, c1^2 + c1 + 1, c1, 'm', 'q');
> sprem(c1^2 + c1 + 1, B1, c1, 'm', 'q');
> factor(%);

Then we solve $R_2 = c_1^2 + c_1 + 1 = 0$.

> B2 := sprem(R2, c1^2 + c1 + 1, c1, 'm', 'q');
> sprem(c1^2 + c1 + 1, B2, c1, 'm', 'q');
> factor(%);

To solve $R_1 = R_2 = c_1^2 - c_1 + 1 = 0$:

We first solve $R_1 = c_1^2 - c_1 + 1 = 0$.

> B3 := sprem(R1, c1^2 - c1 + 1, c1, 'm', 'q');
> sprem(c1^2 - c1 + 1, B3, c1, 'm', 'q');
> factor(%);

Then we solve $R_2 = c_1^2 - c_1 + 1 = 0$.

> B4 := sprem(R2, c1^2 - c1 + 1, c1, 'm', 'q');
> sprem(c1^2 - c1 + 1, B4, c1, 'm', 'q');
> factor(%);

9. References


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN AT MADISON, MADISON, WISCONSIN, 53706
E-mail address: melinda.moran@alumni.nd.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS, 60637
E-mail address: matt_tbo@math.uchicago.edu