MODULI OF MCKAY QUIVER REPRESENTATIONS I: THE COHERENT COMPONENT

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Abstract. For a finite abelian group $G \subset \mathrm{GL}(n, \Bbbk)$, we describe the coherent component Y_θ of the moduli space $\mathcal{M}_\theta$ of θ-stable McKay quiver representations. This is a not-necessarily-normal toric variety that admits a projective birational morphism $Y_\theta \to \mathbb{A}^n_\Bbbk/G$ obtained by variation of GIT quotient. As a special case, this gives a new construction of Nakamura's G-Hilbert scheme Hilb^G that avoids the (typically highly singular) Hilbert scheme of $|G|$-points in $\mathbb{A}^n_\Bbbk$. To conclude, we describe the toric fan of Y_θ and hence calculate the quiver representation corresponding to any point of Y_θ.

1. Introduction

In this paper we concretely describe the coherent component Y_θ of the moduli spaces $\mathcal{M}_\theta$ of θ-stable representations of the McKay quiver for a finite abelian subgroup $G \subset \mathrm{GL}(n, \Bbbk)$ and generic parameter θ, where $\Bbbk$ is an algebraically closed field whose characteristic does not divide $r := |G|$. The irreducible component Y_θ is a not-necessarily-normal toric variety that admits a projective birational morphism $Y_\theta \to \mathbb{A}^n_\Bbbk/G$ obtained by variation of GIT quotient.

The motivation to study the moduli spaces $\mathcal{M}_\theta$ comes from their role in the McKay correspondence (see [BKR01, Hai01, Rei02, BK04]). For a finite subgroup $G \subset \mathrm{SL}(n, \Bbbk)$, this is the expected equivalence between the G-equivariant geometry of $\mathbb{A}^n_\Bbbk$ and the geometry of a crepant resolution $Y \to \mathbb{A}^n_\Bbbk/G$ (if one exists). For $n \leq 3$, Kronheimer [Kro89] and Bridgeland–King–Reid [BKR01] proved that the moduli spaces $\mathcal{M}_\theta$ are crepant resolutions of $\mathbb{A}^n_\Bbbk/G$ for all generic parameters $\theta \in \Theta$. This moduli interpretation of the crepant resolution enabled [BKR01] to establish the McKay correspondence as an equivalence of derived categories for $n \leq 3$. Craw–Ishii [CI04] established a partial converse for finite abelian subgroups $G \subset \mathrm{SL}(3, \Bbbk)$: every projective crepant resolution of $\mathbb{A}^3_\Bbbk/G$ is isomorphic to $\mathcal{M}_\theta$ for some generic parameter $\theta \in \Theta$. For $n \geq 4$, it is unknown whether every projective crepant resolution of $\mathbb{A}^n_\Bbbk/G$ can be constructed as (a component of) $\mathcal{M}_\theta$ for some generic $\theta \in \Theta$. The tools introduced in this paper allow the investigation of such questions. For example, we show in Example 6.4 below that, for the diagonal action of the group $G = \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{A}^n_\Bbbk$ with weights $(1, \ldots, 1)$, the unique toric crepant resolution Y of $\mathbb{A}^n_\Bbbk/G$ is isomorphic to the component Y_θ of $\mathcal{M}_\theta$ for any generic $\theta \in \Theta$.

Since G is abelian, we may assume that G is contained in the subgroup $(\Bbbk^*)^n$ of diagonal matrices with nonzero entries in $\mathrm{GL}(n, \Bbbk)$. This representation of G decomposes into irreducible representations $\rho_1 \oplus \cdots \oplus \rho_n$. The *McKay quiver* is

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the directed graph whose vertices are the irreducible representations ρ of G, with an arrow from $\rho\rho_i$ to ρ for every ρ and $1 \leq i \leq n$. Since G is abelian, there are r vertices and nr arrows. We consider McKay quiver representations of dimension vector $(1, \dots, 1)$, which correspond to points in $\mathbb{A}^{nr}_{\Bbbk}$. Requiring certain commutativity relations gives a subscheme $Z \subset \mathbb{A}^{nr}_{\Bbbk}$. An algebraic torus $T_B = (\Bbbk^*)^r/\Bbbk^*$ acts by change of basis on quiver representations, which in turn gives an action of T_B on Z. Moduli spaces of representations satisfying the commutativity relations are constructed by Geometric Invariant Theory as quotients $\mathcal{M}_\theta := Z /\!\!/_\theta T_B$, where $\theta \in \Theta \cong \mathbb{Q}^{r-1}$ is a fractional character of T_B. The best known example is the G-Hilbert scheme G-Hilb parametrizing *G-clusters* (see [IN00]).

The first main result of this paper constructs explicitly the component Y_θ of $\mathcal{M}_\theta$ that is birational to $\mathbb{A}^n_{\Bbbk}/G$. The crucial step is to introduce an $(r+n) \times nr$-matrix C obtained by augmenting the vertex-edge incidence matrix of the McKay quiver. This matrix defines an irreducible component V of the scheme $Z \subset \mathbb{A}^{nr}_{\Bbbk}$ that has the following properties (see Theorems 3.10 and 4.3).

Theorem 1.1. *The scheme Z has a unique irreducible component $V = \operatorname{Spec} \Bbbk[\mathbb{N}C]$ that does not lie in any coordinate hyperplane of $\mathbb{A}^{nr}_{\Bbbk}$, where $\mathbb{N}C$ is the semigroup generated by the columns of the matrix C. In addition:*

1. *For $\theta \in \Theta$, the GIT quotient $Y_\theta := V /\!\!/_\theta T_B$ is a not-necessarily-normal toric variety that admits a projective birational morphism $\tau_\theta \colon Y_\theta \to \mathbb{A}^n_{\Bbbk}/G$ obtained by variation of GIT quotient.*
2. *For generic $\theta \in \Theta$, the variety Y_θ is the unique irreducible component of $\mathcal{M}_\theta$ containing the T_B-orbit closures of the points of $Z \cap (\Bbbk^*)^{nr}$.*

We call Y_θ the coherent component *of the moduli space $\mathcal{M}_\theta$.*

Corollary 1.2. *For any $\theta \in \Theta$ such that $\mathcal{M}_\theta \cong G$-*Hilb*, the coherent component Y_θ is isomorphic to the irreducible scheme* Hilb^G *introduced by Nakamura.*

This corollary (see Proposition 5.2) provides a direct GIT construction of the scheme Hilb^G that avoids the original construction as a subscheme of the Hilbert scheme of r-points in $\mathbb{A}^n_{\Bbbk}$. In addition, it confirms the suggestion of Mukai that Hilb^G should be obtained from $\mathbb{A}^n_{\Bbbk}/G$ by variation of GIT quotient.

The second main result describes explicitly the set of θ-semistable McKay quiver representations corresponding to points of Y_θ. For generic $\theta \in \Theta$, these representations encode the restriction to Y_θ of the universal quiver representation on the fine moduli space $\mathcal{M}_\theta$ (see Craw–Ishii [CI04, §2]). As a first step, we prove that the toric fan of the variety Y_θ (see Section 6 for details) is the inner normal fan of a polyhedron P_θ obtained by slicing the cone $P \subseteq \mathbb{Q}^{r+n}$ generated by the column vectors of the matrix C (see Corollary 6.2). This explicit description enables us to calculate new examples in detail. In particular, we obtain the new moduli description of the unique crepant toric resolution $Y \to \mathbb{A}^n_{\Bbbk}/G$ for the action of $G = \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{A}^n_{\Bbbk}$ with weights $(1, \dots, 1)$ mentioned above.

Any vector $\mathbf{w} \in (\mathbb{Q}^n_{\geq 0})^\vee$ in the support of the toric fan defining Y_θ determines a distinguished point of Y_θ, and hence a distinguished θ-semistable McKay quiver representation $b_{\theta,\mathbf{w}} \in \mathbb{A}^{nr}_{\Bbbk}$. To compute the coordinates of $b_{\theta,\mathbf{w}}$ explicitly in terms of

$\theta \in \Theta$ and $\mathbf{w} \in (\mathbb{Q}^n_{\geq 0})^\vee$, consider the slice $P^\vee_{\mathbf{w}} := \{\mathbf{v} \in (\mathbb{Q}^r)^\vee : w_i + v_\rho - v_{\rho\rho_i} \geq 0\}$ of the polyhedral cone $P^\vee$ dual to P. The following result is proved in Theorem 7.2.

Theorem 1.3. *Fix $\theta \in \Theta$ and $\mathbf{w} \in (\mathbb{Q}^n_{\geq 0})^\vee$, and let $\mathbf{v} \in P^\vee_{\mathbf{w}}$ be any vector satisfying $\theta \cdot \mathbf{v} \leq \theta \cdot \mathbf{v}'$ for all $\mathbf{v}' \in P^\vee_{\mathbf{w}}$. Then the distinguished θ-semistable quiver representation $b_{\theta,\mathbf{w}} = (b^\rho_i)$ has*

$$b^\rho_i = \begin{cases} 1 & \text{if } w_i + v_\rho - v_{\rho\rho_i} = 0 \\ 0 & \text{if } w_i + v_\rho - v_{\rho\rho_i} > 0 \end{cases} . \tag{1.1}$$

Computing the representations $b_{\theta,\mathbf{w}}$ that give torus-fixed points of Y_θ has been a key tool in understanding the moduli spaces $\mathcal{M}_\theta$ (see [Rei97, Nak01, Cra01, CI04]), though no good algorithm was known in general until now.

From the perspective of string theory, the results of this paper are as follows. For $\Bbbk = \mathbb{C}$, the spaces $\mathcal{M}_\theta$ appear in the physics literature as moduli of $D0$-branes on the orbifold $\mathbb{C}^n/G$, where θ is a Fayet-Iliopoulos term for U(1) gauge multiplets present in the world-volume theory (see [DGM97]). The matrix C introduced in Section 3 encodes both the D-term equations and the F-term equations of the relevant quiver gauge theory. More precisely, the top $r \times nr$ submatrix B encodes the D-terms, giving the moment map for the action of $\mathrm{U}(1)^r/\mathrm{U}(1)$, and the bottom $n \times nr$ submatrix of C encodes the F-terms obtained from the partial derivatives of the superpotential of the quiver gauge theory.

We now explain the division into sections. Section 2 reviews the construction of the moduli spaces $\mathcal{M}_\theta$, including some well-known facts from Geometric Invariant Theory. Section 3 introduces the irreducible component V of Z. Section 4 constructs the coherent component Y_θ to complete the proof of Theorem 1.1, and Section 5 establishes Corollary 1.2. The toric fan of Y_θ is computed in Section 6, and Theorem 1.3 is established in Section 7.

Conventions For an integer matrix C, let $\mathbb{N}C$ denote the semigroup generated by the columns of C. Similarly, $\mathbb{Z}C$ denotes the lattice, $\mathbb{Q}_{\geq 0}C$ the rational cone and $\mathbb{Q}C$ the rational vector space generated by columns of C. A point on a scheme over $\Bbbk$ means a closed point. Write $\Bbbk^*$ for the one-dimensional algebraic torus.

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2. Moduli spaces of McKay quiver representations

In this section we recall the construction of the moduli spaces of representations of the McKay quiver for a finite abelian subgroup $G \subset \mathrm{GL}(n, \Bbbk)$ of order r, where $\Bbbk$ is an algebraically closed field whose characteristic does not divide r. Since G is abelian, we may assume that G is contained in the subgroup $(\Bbbk^*)^n$ of diagonal matrices with nonzero entries in $\mathrm{GL}(n, \Bbbk)$.

Irreducible representations of G are one-dimensional and hence define elements of the dual group of characters $G^* := \operatorname{Hom}(G, \Bbbk^*)$. The n-dimensional representation given by the inclusion of G in $\mathrm{GL}(n, \Bbbk)$ decomposes into one-dimensional representations $\rho_1 \oplus \cdots \oplus \rho_n$ by Schur's lemma, so $g \in G$ acts on $\mathbb{A}^n_{\Bbbk}$ as the diagonal matrix $\operatorname{diag}\bigl(\rho_1(g), \dots, \rho_n(g)\bigr)$. Applying the functor $\operatorname{Hom}(-, \Bbbk^*)$ to this embedding of G into the algebraic torus $T^n \cong (\Bbbk^*)^n$ of diagonal matrices with nonzero entries induces a surjective homomorphism

$$\deg \colon \mathbb{Z}^n \to G^*, \tag{2.1}$$

where $\mathbb{Z}^n = \operatorname{Hom}(T^n, \Bbbk^*)$ is the character lattice of T^n. This determines a G^*-grading of the coordinate ring $\Bbbk[x_1, \dots, x_n]$ of $\mathbb{A}^n_{\Bbbk}$ by setting $\deg(x_i) := \deg(\mathbf{e}_i) = \rho_i$, where $\mathbf{e}_i$ is a standard basis vector of $\mathbb{Z}^n$. Since deg is surjective, $\rho_1, \dots, \rho_n$ generate the group G^*.

Definition 2.1. The *McKay quiver* of $G \subset \mathrm{GL}(n, \Bbbk)$ is the directed graph with a vertex for each $\rho \in G^*$, and an arrow a_i^ρ from $\rho\rho_i$ to ρ for each $\rho \in G^*$ and $1 \leq i \leq n$. We say the arrow a_i^ρ is *labeled* i.

This sign convention agrees with Sardo Infirri [SI96] but differs from that of Craw–Ishii [CI04] where ρ_i is denoted ρ_i^{-1} and the arrows go from ρ to $\rho\rho_i$. Note that the McKay quiver is strongly connected, since for any pair $\rho', \rho \in G^*$ there is a directed path from ρ' to ρ. Every such path comes from writing $\rho^{-1}\rho' = \bigotimes_{1 \leq i \leq n} \rho_i^{m_i}$ for some $m_i \in \mathbb{N}$.

Example 2.2. Consider the subgroup $G := \mathbb{Z}/7\mathbb{Z} \subset \mathrm{GL}(2, \Bbbk)$ generated by the diagonal matrix $g = \operatorname{diag}(\omega, \omega^2)$, where ω is a primitive seventh root of unity. This is the action of type $\frac{1}{7}(1, 2)$. Let $x := x_1$, and $y := x_2$ be the coordinates on $\mathbb{A}^2_{\Bbbk}$. We have $\deg(x) = \rho_1$ and $\deg(y) = \rho_2$, where $\rho_1(g) = \omega$ and $\rho_2(g) = \omega^2$. The McKay quiver has vertices $\rho_0, \rho_1, \dots, \rho_6$ where ρ_0 is the trivial representation of G. Two arrows emanate from each vertex ρ_j, one to $\rho_j\rho_1^{-1} = \rho_{j-1}$ and another to $\rho_j\rho_2^{-1} = \rho_{j-2}$, where addition is modulo 7. The arrows are denoted $a_i^{\rho_j}$ for $i = 1, 2$ and $j = 0, \dots, 6$. The quiver is shown in Figure 1.

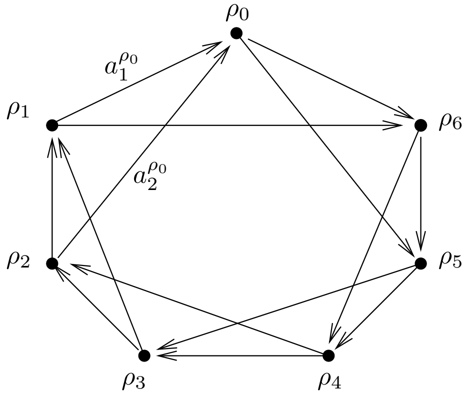

FIGURE 1. The McKay quiver for the action of type $\frac{1}{7}(1, 2)$

Definition 2.3. A *representation of the McKay quiver* with dimension vector $(1,\dots,1) \in \mathbb{N}^r$ is the assignment of a one-dimensional $\Bbbk$-vector space R_ρ to each vertex ρ, and a linear map $R_{\rho\rho_i} \to R_\rho$ to each arrow a_i^ρ in the McKay quiver. Fix a basis for each R_ρ and write $b_i^\rho \in \Bbbk$ for the entry of the 1×1 matrix of the linear map $R_{\rho\rho_i} \to R_\rho$. We occasionally use b_i^ρ to refer to the linear map itself.

Since there are nr arrows in the quiver, representations define points $(b_i^\rho) \in \mathbb{A}^{nr}_\Bbbk$. We write $\Bbbk[z_i^\rho : \rho \in G^*, 1 \le i \le n]$ for the coordinate ring of $\mathbb{A}^{nr}_\Bbbk$. Our interest lies not with the entire space $\mathbb{A}^{nr}_\Bbbk$, but with the points (b_i^ρ) of the scheme Z defined by the ideal

$$I = \langle z_j^{\rho\rho_i} z_i^\rho - z_i^{\rho\rho_j} z_j^\rho : \rho \in G^*, 1 \le i, j \le n\rangle.$$

Thus, we consider only representations (b_i^ρ) satisfying the relations

$$b_j^{\rho\rho_i} b_i^\rho = b_i^{\rho\rho_j} b_j^\rho \text{ for } \rho \in G^* \text{ and } 1 \le i, j \le n \tag{2.2}$$

illustrated in Figure 2. These relations arise naturally when quiver representations are translated into the equivalent language of *G-constellations* (see [CMT05, §2]).

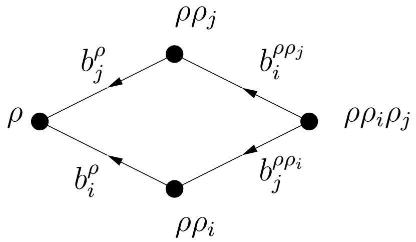

FIGURE 2.

The algebraic torus $\Bbbk^*$ acts on each R_ρ, so $(\Bbbk^*)^r$ acts diagonally on the vector space $\oplus_{\rho\in G^*} R_\rho$. Hence $t = (t_\rho) \in (\Bbbk^*)^r$ acts on $b_i^\rho \in \operatorname{Hom}(R_{\rho\rho_i}, R_\rho) = R^*_{\rho\rho_i} \otimes R_\rho$ as

$$t \cdot b_i^\rho = t^{-1}_{\rho\rho_i} t_\rho b_i^\rho. \tag{2.3}$$

This is a change of basis, with the new basis vector of R_ρ set to be t_ρ^{-1} times the old one. We now describe this action in terms of a matrix. Write $\{\mathbf{e}_i^\rho : \rho \in G^*, 1 \le i \le n\}$ for the standard basis of $\mathbb{Z}^{nr}$. Order the basis globally into r blocks, one for each $\rho \in G^*$ beginning with the trivial representation ρ_0. Within each block the elements are listed $\mathbf{e}_1^\rho, \dots, \mathbf{e}_n^\rho$. Let B be the $r \times (nr)$ matrix with the column corresponding to $\mathbf{e}_i^\rho$ being $\mathbf{e}_\rho - \mathbf{e}_{\rho\rho_i}$. Let $\mathbb{N}B \subset \mathbb{Z}^r$ be the semigroup generated by the columns of B. The matrix B is the vertex-edge incidence matrix of the McKay quiver. The columns of B encode the weights of the action defined in (2.3).

Lemma 2.4. *The subsemigroup $\mathbb{N}B \subset \mathbb{Z}^r$ coincides with the sublattice of $\mathbb{Z}^r$ given by $\mathbf{1}^\perp := \{(\theta_\rho) \in \mathbb{Z}^r : \sum_\rho \theta_\rho = 0\}$.*

Proof. The columns of B lie in $\mathbf{1}^\perp$, so $\mathbb{N}B \subseteq \mathbf{1}^\perp$. For the opposite inclusion, suppose there is some $\theta \in \mathbf{1}^\perp \smallsetminus \mathbb{N}B$. We may assume that θ has $\sum_\rho |\theta_\rho|$ minimal for such a vector. Pick ρ with $\theta_\rho < 0$ and ρ' with $\theta_{\rho'} > 0$. The sum of the columns of B corresponding to the arrows in a directed path from ρ to ρ' is the vector $\mathbf{e}_{\rho'} - \mathbf{e}_\rho$. Now $\theta' := \theta + \mathbf{e}_\rho - \mathbf{e}_{\rho'} \in \mathbf{1}^\perp$, and $\sum_\rho |\theta'_\rho|$ is smaller, so $\theta' \in \mathbb{N}B$, and thus also $\theta \in \mathbb{N}B$. □

By Lemma 2.4, the sublattice $\mathbb{Z}B \subset \mathbb{Z}^r$ generated by the columns of B equals the semigroup $\mathbb{N}B$. The matrix B determines an action of the algebraic torus $T_B := \operatorname{Hom}(\mathbb{Z}B, \Bbbk^*)$ on $\mathbb{A}^{nr}_\Bbbk$ by formula (2.3) for $t = (t_\rho) \in T_B$. Note that $T_B = (\Bbbk^*)^r/\Bbbk^*$. The ideal I defining Z is invariant under this action, so T_B acts on Z.

The n-dimensional torus T^n of diagonal matrices with nonzero entries acting on $\mathbb{A}^n_\Bbbk$ also acts on $\mathbb{A}^{nr}_\Bbbk$, where $s = (s_i) \in T^n$ acts on (b^ρ_i) as

$$s \cdot (b^\rho_i) = (s_i b^\rho_i). \tag{2.4}$$

Again, the ideal I defining Z is invariant under this action, so T^n acts on Z.

To define moduli spaces of McKay quiver representations we consider equivalence classes of quiver representations $(b^\rho_i) \in Z$ modulo the T_B-action. To construct the quotients we use Geometric Invariant Theory (GIT). For convenience we recall the general construction of the GIT quotient of an affine scheme $X \subseteq \mathbb{A}^d_\Bbbk$ by the linear action of an algebraic torus $T \cong (\Bbbk^*)^s$ (see Dolgachev [Dol03]).

Choose coordinates on $\mathbb{A}^d_\Bbbk$ to diagonalize the T-action, so for $v = (v_1, \dots, v_d) \in \mathbb{A}^d_\Bbbk$ we have $t \cdot v_i = t^{\mathbf{a}_i} v_i$ for some character $\mathbf{a}_i \in T^* = \operatorname{Hom}(T, \Bbbk^*) \cong \mathbb{Z}^s$ and $t \in T$. This gives a $\mathbb{Z}^s$-grading of $\Bbbk[z_1, \dots, z_d]$ by setting $\deg(z_i) = \mathbf{a}_i$. Since T acts on X, the defining ideal of X in $\Bbbk[z_1, \dots, z_d]$ is homogeneous, and thus $\Bbbk[X]$ is $\mathbb{Z}^s$-graded. Line bundles on X are trivial, so a linearization is a lift of the T-action from X to $X \times \mathbb{A}^1_\Bbbk$. If we write $\Bbbk[\lambda]$ for the coordinate ring of $\mathbb{A}^1_\Bbbk$, then any such lift is determined by $t \cdot \lambda = t^{-\mathbf{b}}\lambda$ for some character $\mathbf{b} \in T^*$. Then for $f \in \Bbbk[X]$ and $j > 0$, the function $f\lambda^j \in \Bbbk[X \times \mathbb{A}^1_\Bbbk]$ is T-invariant if and only if $\deg(f) = j\mathbf{b}$. Let $\Bbbk[X]_{j\mathbf{b}}$ be the $j\mathbf{b}$-graded piece of $\Bbbk[X]$. Then the GIT quotient of X by the action of T linearized by $\mathbf{b}$ is

$$X /\!\!/_{\mathbf{b}} T := \operatorname{Proj} \textstyle\bigoplus_{j \geq 0} \Bbbk[X]_{j\mathbf{b}}.$$

This scheme is the categorical quotient of the open subscheme of X consisting of $\mathbf{b}$-semistable points. Recall that a point $x \in X$ is $\mathbf{b}$-semistable if there exists $j > 0$ and $s \in \Bbbk[X]_{j\mathbf{b}}$ such that $s(x) \neq 0$.

Since $\Bbbk[X]_{\mathbf{0}}$ is a subalgebra of the graded ring defining $X /\!\!/_{\mathbf{b}} T$, the Proj construction induces a projective morphism from $X /\!\!/_{\mathbf{b}} T$ to the quotient $X /\!\!/_{\mathbf{0}} T = \operatorname{Spec} \Bbbk[X]^T$ linearized by the trivial character. Moreover, the line bundle $\mathcal{O}(1)$ coming from the Proj construction is relatively ample with respect to this morphism. As is standard in GIT, the quotient linearized by a *fractional character* $\mathbf{b} \in T^* \otimes \mathbb{Q}$ is defined to be the GIT quotient linearized by any multiple that gives an integral character $j\mathbf{b} \in T^*$ (the quotient carries a bundle $\mathcal{O}(1)$ for which $\mathcal{O}(j)$ is relatively ample). A character $\mathbf{b} \in T^* \otimes \mathbb{Q}$ is *generic* if every $\mathbf{b}$-semistable point of X is in fact $\mathbf{b}$-stable, in which case the categorical quotient $X /\!\!/_{\mathbf{b}} T$ is a geometric quotient. Recall that a $\mathbf{b}$-semistable point is $\mathbf{b}$-stable if the dimension of the stabilizer T_x is finite and if there exists some $s \in \Bbbk[X]_{j\mathbf{b}}$ as above for which the T-action on the set $\{y \in X : s(y) \neq 0\}$ is closed.

We return to the case of interest, where $T_B = \operatorname{Hom}(\mathbb{Z}B, \Bbbk^*)$ acts on the affine scheme Z as in (2.3). In this case, $T_B^* \otimes \mathbb{Q}$ is the vector space $\mathbb{Q}B := \mathbb{Z}B \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the columns of B. By Lemma 2.4 we may regard $\mathbb{Q}B$ as a hyperplane in the space of fractional characters of the original $(\Bbbk^*)^r$-action as follows.

Definition 2.5. The *GIT parameter space* is the $\mathbb{Q}$-vector space

$$\Theta := \mathbb{Z}B \otimes \mathbb{Q} = \big\{(\theta_\rho) \in \mathbb{Q}^r : \textstyle\sum_{\rho \in G^*} \theta_\rho = 0\big\}.$$

For $\theta \in \Theta$, the scheme $\mathcal{M}_\theta := Z /\!\!/_\theta T_B$ is the *moduli space of θ-semistable McKay quiver representations* of dimension vector $(1, \ldots, 1)$ satisfying the relations (2.2). For generic θ, $\mathcal{M}_\theta$ is the *fine moduli space of θ-stable McKay quiver representations.*

Remark 2.6. For a finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$, Kronheimer [Kro89] proved that the geometric quotient $Z /\!\!/_\theta T$ coincides with the minimal resolution of $\mathbb{A}^2_{\mathbb{C}}/G$ for generic $\theta \in \Theta$. The method introduced by Ishii [Ish02] extends this result to any finite subgroup of $\mathrm{GL}(2, \Bbbk)$.

3. A DISTINGUISHED COMPONENT OF Z

This section identifies a distinguished irreducible component V of the scheme Z introduced in Section 2 and proves that V is a not-necessarily-normal toric variety.

Note first that Z need not be irreducible.

Example 3.1. For the action of type $\frac{1}{7}(1, 2)$ from Example 2.2, the scheme Z is defined by the ideal

$$\begin{aligned} I = \langle & z_2^{\rho_1} z_1^{\rho_0} - z_1^{\rho_2} z_2^{\rho_0}, z_2^{\rho_2} z_1^{\rho_1} - z_1^{\rho_3} z_2^{\rho_1}, z_2^{\rho_3} z_1^{\rho_2} - z_1^{\rho_4} z_2^{\rho_2}, z_2^{\rho_4} z_1^{\rho_3} - z_1^{\rho_5} z_2^{\rho_3}, \\ & z_2^{\rho_5} z_1^{\rho_4} - z_1^{\rho_6} z_2^{\rho_4}, z_2^{\rho_6} z_1^{\rho_5} - z_1^{\rho_0} z_2^{\rho_5}, z_2^{\rho_0} z_1^{\rho_6} - z_1^{\rho_1} z_2^{\rho_6} \rangle. \end{aligned}$$

The ideal I has eight associated primes and hence Z is reducible. One of the associated primes is

$$\begin{aligned} J_1 = \langle & z_1^{\rho_0} z_2^{\rho_5} - z_1^{\rho_5} z_2^{\rho_6}, z_1^{\rho_6} z_2^{\rho_4} - z_1^{\rho_4} z_2^{\rho_5}, z_1^{\rho_5} z_2^{\rho_3} - z_1^{\rho_3} z_2^{\rho_4}, z_1^{\rho_4} z_2^{\rho_2} - z_1^{\rho_2} z_2^{\rho_3}, \\ & z_1^{\rho_3} z_2^{\rho_1} - z_1^{\rho_1} z_2^{\rho_2}, z_1^{\rho_6} z_2^{\rho_0} - z_1^{\rho_1} z_2^{\rho_6}, z_1^{\rho_2} z_2^{\rho_0} - z_1^{\rho_0} z_2^{\rho_1}, z_1^{\rho_0} z_1^{\rho_6} z_2^{\rho_3} - z_1^{\rho_3} z_1^{\rho_4} z_2^{\rho_6}, \\ & z_1^{\rho_5} z_1^{\rho_6} z_2^{\rho_2} - z_1^{\rho_2} z_1^{\rho_3} z_2^{\rho_5}, z_1^{\rho_0} z_1^{\rho_6} z_2^{\rho_2} - z_1^{\rho_2} z_1^{\rho_3} z_2^{\rho_6}, z_1^{\rho_5} z_1^{\rho_6} z_2^{\rho_1} - z_1^{\rho_1} z_1^{\rho_2} z_2^{\rho_5}, \\ & z_1^{\rho_4} z_1^{\rho_5} z_2^{\rho_1} - z_1^{\rho_1} z_1^{\rho_2} z_2^{\rho_4}, z_1^{\rho_4} z_1^{\rho_5} z_2^{\rho_0} - z_1^{\rho_0} z_1^{\rho_1} z_2^{\rho_4}, z_1^{\rho_3} z_1^{\rho_4} z_2^{\rho_0} - z_1^{\rho_0} z_1^{\rho_1} z_2^{\rho_3} \rangle. \end{aligned}$$

The distinguished component V of Z is the variety defined by J_1. See Theorem 3.10 for details. Another associated prime is

$$J_2 = \langle z_1^{\rho_1}, z_2^{\rho_1}, z_2^{\rho_2}, z_1^{\rho_5}, z_2^{\rho_5}, z_2^{\rho_6}, z_1^{\rho_4} z_2^{\rho_3} - z_2^{\rho_4} z_1^{\rho_2} \rangle.$$

The remaining six are obtained by adding $j \mod 7$ to every raised index in J_2, for $j = 1, \ldots, 6$. The associated primes of I were computed using the computer algebra package Macaulay 2 [GS].

We now present a sequence of combinatorial lemmas that enable us to define V explicitly in Theorem 3.10. Let $\{\mathbf{e}_\rho : \rho \in G^*\} \cup \{\mathbf{e}_i : 1 \leq i \leq n\}$ be the standard basis of $\mathbb{Z}^{r+n}$. Denote by $\pi_n \colon \mathbb{Z}^{r+n} \to \mathbb{Z}^n$ the projection to the last n coordinates.

Definition 3.2. Let C be the $(r + n) \times nr$ matrix whose top r rows form the vertex-edge incidence matrix B and whose bottom n rows record the label of the corresponding arrow. Specifically, the column of C indexed by the arrow a_i^ρ is $C_i^\rho := \mathbf{e}_\rho - \mathbf{e}_{\rho\rho_i} + \mathbf{e}_i$.

Example 3.3. For the action of type $\frac{1}{7}(1,2)$ from Example 2.2, C is the 9×14 matrix shown below and B is the top 7×14 submatrix. The third column corresponds to the arrow $a_1^{\rho_1}$ labeled $i = 1$ with tail at ρ_2 and head at ρ_1.

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

An undirected path (cycle) in the McKay quiver is a path (cycle) in the underlying undirected graph. Since all arrows in the McKay quiver are directed, there may be some arrows in the path that are traversed according to their orientation, and some against. We write $-a_i^\rho$ for the arrow a_i^ρ traversed against its orientation.

Definition 3.4. Let γ be an undirected path in the McKay quiver. The *vector* of γ, denoted $\mathbf{v}(\gamma) \in \mathbb{Z}^{nr}$, is defined by setting $\mathbf{v}(\gamma)_i^\rho$ to be the number of times the arrow a_i^ρ is traversed according to its orientation in the McKay quiver minus the number of times it is traversed against its orientation. The *type* of a path γ is $\pi_n(C\mathbf{v}(\gamma)) \in \mathbb{Z}^n$. The type records the number of arrows of each label, where an arrow is counted as negative if it is traversed against its orientation. Of particular importance are paths of type $\mathbf{0} \in \mathbb{Z}^n$.

Example 3.5. Consider the McKay quiver for $\frac{1}{7}(1,2)$ shown in Example 2.2. The path γ consisting of the arrows $a_1^{\rho_0}, a_1^{\rho_6}, -a_2^{\rho_6}, -a_1^{\rho_1}$, in that order, has vector $\mathbf{v}(\gamma) = (1,0,-1,0,0,0,0,0,0,0,0,0,1,-1)$ and type $(1,-1)$.

Lemma 3.6. *A vector* $\mathbf{u} \in \ker_{\mathbb{Z}}(B) := \{\mathbf{u} \in \mathbb{Z}^{nr} : B\mathbf{u} = \mathbf{0}\}$ *if and only if there is an undirected cycle* γ *in the McKay quiver with vector* $\mathbf{v}(\gamma) = \mathbf{u}$.

Proof. Exercise 38 of Bollobás [Bol98, II.3] implies that a vector $\mathbf{u}$ lies in $\ker_{\mathbb{Z}}(B)$ if and only if there is a collection $\{\gamma_i\}$ of undirected cycles in the McKay quiver with $\mathbf{u} = \sum_i \mathbf{v}(\gamma_i)$. Since the McKay quiver is connected, each γ_i can be connected to a base vertex via a path that is traversed once in the forward direction, and once in the reverse direction. The vector of this augmented cycle equals $\mathbf{v}(\gamma_i)$. Attaching all cycles to this base vertex produces a single cycle γ with vector $\mathbf{v}(\gamma) = \mathbf{u}$. □

Definition 3.7. For $1 \leq i, j \leq n$ with $i \neq j$, and $\rho \in G^*$, define

$$\mathbf{c}_{i,j}^\rho := \mathbf{e}_i^\rho + \mathbf{e}_j^{\rho\rho_i} - \mathbf{e}_j^\rho - \mathbf{e}_i^{\rho\rho_j} \in \mathbb{Z}^{nr}.$$

The vectors $\mathbf{c}_{i,j}^\rho$ lie in $\ker_{\mathbb{Z}}(B)$.

Lemma 3.8. *Let* γ *be a path with two adjacent arrows labeled* i *and* j*, with* $i \neq j$*. Then there is a path* γ' *satisfying* $\mathbf{v}(\gamma) - \mathbf{v}(\gamma') = \pm\mathbf{c}_{i,j}^\rho$ *that differs from* γ *only by replacing the pair of arrows labeled* i *and* j *with a pair labeled* j *and* i *respectively.*

Proof. The choice of the replacement arrows depends on the orientation of the arrows with labels i and j, and divides into four cases.

If the arrows labeled i and j meeting at ρ are both traversed according to their orientation, replace the arrows $a_i^{\rho}, a_j^{\rho\rho_j^{-1}}$ by $a_j^{\rho\rho_i\rho_j^{-1}}, a_i^{\rho\rho_j^{-1}}$ using $\mathbf{c}_{i,j}^{\rho\rho_j^{-1}}$ as shown in Figure 3(i). In this figure the paths go from right to left, and we replace the path at the top of the diamond by the one at the bottom.

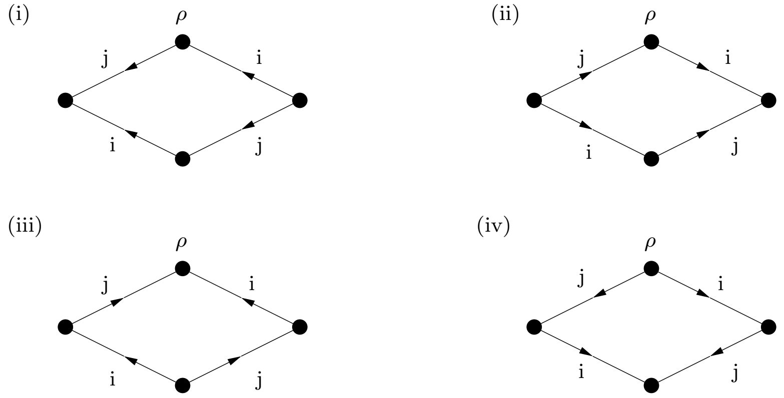

Figure 3.

If the arrows labeled i and j meeting at ρ are both traversed against their orientation, replace the arrows $a_i^{\rho\rho_i^{-1}}, a_j^{\rho}$ by $a_j^{\rho\rho_i^{-1}}, a_i^{\rho\rho_j\rho_i^{-1}}$ using $\mathbf{c}_{i,j}^{\rho\rho_i^{-1}}$ as in Figure 3(ii).

If the arrow labeled i is traversed according to its orientation, but the arrow labeled j against, replace a_i^{ρ}, a_j^{ρ} by $a_j^{\rho\rho_i}, a_i^{\rho\rho_j}$ using $\mathbf{c}_{i,j}^{\rho}$ as in Figure 3(iii).

If the arrow labeled j is traversed according to its orientation, but the arrow labeled i against, replace $a_i^{\rho\rho_i^{-1}}, a_j^{\rho\rho_j^{-1}}$ by $a_j^{\rho\rho_i^{-1}\rho_j^{-1}}, a_i^{\rho\rho_i^{-1}\rho_j^{-1}}$ using $\mathbf{c}_{i,j}^{\rho\rho_i^{-1}\rho_j^{-1}}$ as in Figure 3(iv).

In each case, the orientation of each type of arrow is preserved while the labeling of the arrows is switched. ☐

The following result, whose proof requires Lemmas 3.6 and 3.8, is the key ingredient in Theorem 3.10.

Lemma 3.9. *The set* $\{\mathbf{c}_{i,j}^{\rho} : 1 \leq i, j \leq n, i \neq j, \rho \in G^*\}$ *of* $r\binom{n}{2}$ *vectors in* $\mathbb{Z}^{nr}$ *generates the lattice* $\ker_{\mathbb{Z}}(C) = \{\mathbf{u} \in \mathbb{Z}^{nr} : C\mathbf{u} = \mathbf{0}\}$.

Proof. Let $\mathbf{u} \in \ker_{\mathbb{Z}}(C)$. We need to show that $\mathbf{u}$ can be written as a $\mathbb{Z}$-linear combination of the $\mathbf{c}_{i,j}^{\rho}$. By Lemma 3.6 there is a cycle γ in the McKay quiver with $\mathbf{v}(\gamma) = \mathbf{u}$. Since $\mathbf{u} \in \ker_{\mathbb{Z}}(C)$, γ has type $\mathbf{0}$. Suppose now that $\mathbf{u}$ does not lie in the integer span of the $\mathbf{c}_{i,j}^{\rho}$. We may assume that any cycle γ' consisting of fewer arrows than γ has $\mathbf{v}(\gamma')$ in the integer span of $\mathbf{c}_{i,j}^{\rho}$.

Consider an arrow in γ labeled i. By Lemma 3.8 we can find a cycle γ' with $C\mathbf{v}(\gamma') = C\mathbf{v}(\gamma)$ where $\mathbf{v}(\gamma) - \mathbf{v}(\gamma')$ is (up to sign) one of the $\mathbf{c}_{i,j}^{\rho}$, and the arrow labeled i has been switched with an adjacent arrow labeled j. By repeatedly using Lemma 3.8 we can move this arrow around until it is adjacent to another arrow labeled i. If this new arrow labeled i is traversed in the same direction, repeat with this new arrow. Eventually an arrow labeled i traversed in the opposite direction must occur, since γ has type $\mathbf{0}$. But then we have a cycle γ'' with two adjacent arrows with the same labels but traversed in opposite directions. Either these two arrows labeled i have their heads in a common vertex ρ or their tails in a common vertex ρ. Since every vertex in the McKay quiver has only one incoming and one outgoing edge of the same label, the above situation can happen if and only if the cycle reaches a vertex using the arrow labeled i and then traverses the same arrow in the opposite direction. Removing this cycle of length two produces a cycle γ''' satisfying $\mathbf{v}(\gamma''') = \mathbf{v}(\gamma'')$. Since γ''' consists of fewer arrows than γ, the vector $\mathbf{v}(\gamma''')$ lies in the integer span of the $\mathbf{c}^{\rho}_{i,j}$ by assumption. The difference $\mathbf{v}(\gamma) - \mathbf{v}(\gamma''')$ also lies in the integer span of the $\mathbf{c}^{\rho}_{i,j}$, so $\mathbf{v}(\gamma)$ lies in the integer span of the $\mathbf{c}^{\rho}_{i,j}$ after all. □

For $\mathbf{u} = (u^{\rho}_i) \in \mathbb{N}^{nr}$, we write $z^{\mathbf{u}}$ for the monomial in the polynomial ring $\Bbbk[z^{\rho}_i : \rho \in G^*, 1 \leq i \leq n]$ that is the product over all i and ρ of z^{ρ}_i raised to the power u^{ρ}_i. The *toric ideal* of the matrix C is the ideal

$$I_C := \langle z^{\mathbf{u}} - z^{\mathbf{v}} \,:\, \mathbf{u}, \mathbf{v} \in \mathbb{N}^{nr},\ \mathbf{u} - \mathbf{v} \in \ker_{\mathbb{Z}}(C) \rangle$$

in $\Bbbk[z^{\rho}_i : \rho \in G^*, 1 \leq i \leq n]$. As Sturmfels [Stu96, §4,13] describes, the toric ideal I_C defines the not-necessarily-normal affine toric variety $\operatorname{Spec} \Bbbk[\mathbb{N}C]$.

Theorem 3.10. *There is a unique irreducible component of Z that does not lie on any coordinate hyperplane in $\mathbb{A}^{nr}_{\Bbbk}$. This component is the affine variety $V :=$ $\operatorname{Spec} \Bbbk[\mathbb{N}C]$ defined by the toric ideal I_C, and is thus reduced.*

Proof. By Proposition 3.9 the vectors $\mathbf{c}^{\rho}_{i,j}$ generate the lattice $\ker_{\mathbb{Z}}(C)$. They are precisely the differences of exponents of the generators of the binomial ideal I defining Z. Hoşten–Sturmfels [HS95] show that if $\{\mathbf{u}_i - \mathbf{v}_i : i = 1, \dots, t\}$ generates the lattice $\ker_{\mathbb{Z}}(C)$ with $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{N}^{nr}$ and $J_C := \langle z^{\mathbf{u}_i} - z^{\mathbf{v}_i} : i = 1, \dots, t \rangle$, then the toric ideal $I_C = (J_C : (\prod z^{\rho}_i)^{\infty})$. Hence in our case we have

$$I_C = (I : (\prod z^{\rho}_i)^{\infty}).$$

The saturation of I by the product of all variables is the ideal of the intersection of the components of Z not lying in any coordinate hyperplane. Since I_C is prime, its variety is therefore the unique component of Z that does not lie entirely in any coordinate hyperplane. This is the variety $V = \operatorname{Spec} \Bbbk[\mathbb{N}C]$, since $\Bbbk[\mathbb{N}C] = \Bbbk[\mathbb{A}^{nr}_{\Bbbk}]/I_C$. Since I_C is prime, V is reduced and irreducible. □

Remark 3.11. Sardo Infirri [SI96, Proposition 5.3] claimed that Z is the (irreducible) toric variety $\operatorname{Spec} \Bbbk[\mathbb{N}C]$. In Example 3.1 we saw that Z may be reducible. In addition, he assumed that $\mathbb{N}C = \mathbb{Q}_{\geq 0}C \cap \mathbb{Z}C$ without proof, from which normality of $V = \operatorname{Spec} \Bbbk[\mathbb{N}C]$ would follow. We show in [CMT05] that V is not always normal.

Remark 3.12. The columns of the matrix C encode the weights of the product $T_B \times T^n$ of the torus actions on $\mathbb{A}^{nr}_{\Bbbk}$ given in (2.3) and (2.4). Thus, the action of the dense torus $T_C = \operatorname{Hom}(\mathbb{Z}C, \Bbbk^*)$ on V is equal to the restriction to $V \subset \mathbb{A}^{nr}_{\Bbbk}$ of the $T_B \times T^n$-action on $\mathbb{A}^{nr}_{\Bbbk}$.

4. GIT construction of the coherent component

In this section we show that for all $\theta \in \Theta$, the GIT quotient $Y_\theta := V /\!\!/_\theta T_B$ is a not-necessarily-normal toric variety that is obtained from $\mathbb{A}^n_{\Bbbk}/G$ by variation of GIT quotient.

We begin with the construction of $V /\!\!/_\theta T_B$. The action of the torus T_B on $\mathbb{A}^{nr}_{\Bbbk}$ gives a $\mathbb{Z}B$-grading of $\Bbbk[\mathbb{A}^{nr}_{\Bbbk}]$ by $\deg(z^\rho_i) = \mathbf{e}_\rho - \mathbf{e}_{\rho\rho_i}$. Since I_C is homogeneous in this grading, we obtain a grading of the coordinate ring $\Bbbk[V]$. Write $\pi\colon \mathbb{Z}C \to \mathbb{Z}B$ for the restriction of the projection $\pi_r\colon \mathbb{Z}^{r+n} \to \mathbb{Z}^r$ onto the first r coordinates. Since $\Bbbk[V] = \Bbbk[\mathbb{N}C]$, the $\mathbb{Z}B$-grading on $\Bbbk[V]$ is induced by π. In particular, for any element $\theta \in \Theta$ in the GIT parameter space (see Definition 2.5), the θ-graded piece $\Bbbk[V]_\theta$ is the $\Bbbk$-vector space spanned by the monomials whose exponents lie in $\mathbb{N}C \cap \pi^{-1}(\theta)$. Note that this set is nonempty for $\theta \in \mathbb{Z}B$ by Lemma 2.4. As a result, the categorical quotient of V by the action of T_B linearized by $\theta \in \Theta$ is

$$V /\!\!/_\theta T_B = \operatorname{Proj} \textstyle\bigoplus_{j\geq 0} \Bbbk[V]_{j\theta} \tag{4.1}$$

where, as before, the definition for a fractional character $\theta \in \Theta$ is taken to mean $V /\!\!/_{j\theta} T_B$ for some $j \in \mathbb{N}$ satisfying $j\theta \in \mathbb{Z}B$.

Let $M \subset \mathbb{Z}^n$ be the kernel of the group homomorphism $\deg\colon \mathbb{Z}^n \to G^*$ from equation (2.1). Observe that $\mathbb{A}^n_{\Bbbk}/G = \operatorname{Spec} \Bbbk[\mathbb{A}^n_{\Bbbk}]^G = \operatorname{Spec} \Bbbk[\mathbb{N}^n \cap M]$.

Proposition 4.1. *The categorical quotient $V /\!\!/_{\mathbf{0}} T_B$ is isomorphic to $\mathbb{A}^n_{\Bbbk}/G$.*

Proof. For $\theta = \mathbf{0}$ we have $V /\!\!/_{\mathbf{0}} T_B = \operatorname{Spec} \Bbbk[V]_{\mathbf{0}}$. Since $\mathbb{A}^n_{\Bbbk}/G = \operatorname{Spec} \Bbbk[\mathbb{N}^n \cap M]$, the proposition follows once we identify $\mathbb{N}C \cap \ker_{\mathbb{Z}}(\pi)$ with the semigroup $\mathbb{N}^n \cap M$.

The first step is to show that $\pi_n\colon \mathbb{Z}^{r+n} \to \mathbb{Z}^n$ induces a lattice isomorphism between $\ker_{\mathbb{Z}}(\pi)$ and M. This is equivalent to showing that the respective tori are the same. The lattice $\ker_{\mathbb{Z}}(\pi)$ is a sublattice of $\ker_{\mathbb{Z}}(\pi_r)$ and $\pi_n(\ker_{\mathbb{Z}}(\pi)) \subseteq \pi_n(\ker_{\mathbb{Z}}(\pi_r)) = \mathbb{Z}^n$. To see that $\pi_n(\ker_{\mathbb{Z}}(\pi)) \subseteq M$, consider $C\mathbf{u} \in \ker_{\mathbb{Z}}(\pi)$ for $\mathbf{u} \in \mathbb{Z}^{nr}$. Since $\mathbf{u} \in \ker_{\mathbb{Z}}(B)$, Lemma 3.6 produces a cycle γ in the McKay quiver with $\mathbf{v}(\gamma) = \mathbf{u}$. The type of a path from ρ' to ρ in the McKay quiver is an element of $\mathbb{Z}^n$ of degree $\rho^{-1}\rho'$, so a path is a cycle if and only if its type lies in M. Thus $\pi_n(C\mathbf{v}(\gamma))$ lies in the sublattice $M \subset \mathbb{Z}^n$. This gives $\pi_n(C\mathbf{u}) \in M$ as claimed. For the opposite inclusion, choose $\mathbf{m} = (m_1, \dots, m_n) \in M$ and construct the path γ in the McKay quiver from any vertex ρ beginning with the connected sequence of $|m_1|$ arrows labeled 1, oriented according to the sign of m_1, followed by the connected sequence of $|m_2|$ arrows labeled 2, oriented according to the sign of m_2, and so on. This path has type $\mathbf{m}$ by construction, so γ is a cycle, and thus $\mathbf{v}(\gamma) \in \ker_{\mathbb{Z}}(B)$. This gives $C\mathbf{v}(\gamma) \in \ker_{\mathbb{Z}}(\pi)$, so $\pi_n(C\mathbf{v}(\gamma)) \in \pi_n(\ker_{\mathbb{Z}}(\pi))$ and hence $M \subseteq \pi_n(\ker_{\mathbb{Z}}(\pi))$. Since the restriction of π_n to $\ker_{\mathbb{Z}}(\pi)$ is an isomorphism, the claim follows.

It remains to show that $\pi_n(\mathbb{N}C \cap \ker_{\mathbb{Z}}(\pi)) = \mathbb{N}^n \cap M$. Since $\mathbb{N}C$ is generated by vectors of the form $\mathbf{e}_\rho - \mathbf{e}_{\rho\rho_i} + \mathbf{e}_i$, the semigroup $\pi_n(\mathbb{N}C)$ lies in the subsemigroup $\mathbb{N}^n \subset \mathbb{Z}^n$ generated by the elements $\pi_n(\mathbf{e}_\rho - \mathbf{e}_{\rho\rho_i} + \mathbf{e}_i) = \mathbf{e}_i$ for $1 \leq i \leq n$. Combining this with the above gives $\pi_n(\mathbb{N}C \cap \ker_{\mathbb{Z}}(\pi)) \subseteq \mathbb{N}^n \cap M$. To establish equality, observe that in the proof of the inclusion $M \subseteq \pi_n(\ker_{\mathbb{Z}}(\pi))$ described above, if each entry of $\mathbf{m} \in M$ is nonnegative then the path γ has $\mathbf{v}(\gamma) \in \mathbb{N}^{nr}$. This gives an element $C\mathbf{v}(\gamma) \in \mathbb{N}C \cap \ker_{\mathbb{Z}}(\pi)$ satisfying $\pi_n(\mathbf{v}(\gamma)) \in \mathbb{N}^n \cap M$. $\square$

Remark 4.2. The second paragraph of the proof of Proposition 4.1 shows that the torus $T_C /\!\!/_{\mathbf{0}} T_B = \operatorname{Spec} \Bbbk[\mathbb{Z}C]^{T_B} = \operatorname{Spec} \Bbbk[\ker_{\mathbb{Z}}(\pi)]$ is isomorphic to the standard torus $T^n/G = \operatorname{Spec} \Bbbk[M]$ of the toric variety $\mathbb{A}^n_{\Bbbk}/G$.

Theorem 4.3. *For $\theta \in \Theta$, set $Y_\theta := V /\!\!/_\theta T_B$. Then:*

- (i) *Y_θ is a not-necessarily-normal toric variety that admits a projective birational morphism $\tau_\theta \colon Y_\theta \to \mathbb{A}^n_{\Bbbk}/G$ obtained by variation of GIT quotient.*
- (ii) *For generic $\theta \in \Theta$, the variety Y_θ is the unique irreducible component of the moduli space $\mathcal{M}_\theta$ containing the T_B-orbit closures of points of $Z \cap (\Bbbk^*)^{nr}$.*

For generic $\theta \in \Theta$, we call Y_θ the coherent component *of $\mathcal{M}_\theta$.*

Proof. The composition of the canonical projective morphism $V /\!\!/_\theta T_B \to V /\!\!/_{\mathbf{0}} T_B$ described in Section 2 with the isomorphism from Proposition 4.1 gives the projective morphism τ_θ for all $\theta \in \Theta$. To prove that τ_θ is birational we first prove that $T_C /\!\!/_\theta T_B$ is isomorphic to T^n/G. Since T_C/T_B is isomorphic to T^n/G by Remark 4.2, we need only show that $T_C /\!\!/_\theta T_B$ is isomorphic to T_C/T_B for all $\theta \in \Theta$ or, equivalently, that every point of T_C is θ-semistable. To see this, note that each monomial $\mathbf{x}^{\mathbf{u}} \in \Bbbk[V]_\theta$ is nonzero on every point of T_C because the coordinate entries of every point of T_C are nonzero under the given embedding $T_C \subset (\Bbbk^*)^{nr}$. This shows that T_C is θ-semistable, so τ_θ is birational.

To complete the proof of the first statement we show that $Y_\theta = T_C /\!\!/_\theta T_B$ is a not-necessarily-normal toric variety. The universal property of categorical quotients (see [MFK94, §0.2]) ensures that Y_θ inherits reducedness and irreducibility from V. Since $T_C \subset V$ is dense, and since $T_C /\!\!/_\theta T_B$ is nonempty by the above, the torus $T_C /\!\!/_\theta T_B$ is dense in Y_θ. Moreover, the action of T_C on V descends to an action of $T_C /\!\!/_\theta T_B$ on Y_θ as required.

To prove the second statement, let $\theta \in \Theta$ be generic, so $V /\!\!/_\theta T_B$ is a geometric quotient. The inclusion of V in Z induces an inclusion of Y_θ into $\mathcal{M}_\theta$. Since Y_θ is reduced and irreducible, it is a component of $\mathcal{M}_\theta$ unless there is some irreducible component $W \subset Z$ such that $W /\!\!/_\theta T_B$ contains $V /\!\!/_\theta T_B$ as a proper closed subscheme. Let V^s_θ and W^s_θ denote the loci of θ-stable points of V and W respectively, and write $p \colon W^s_\theta \to W /\!\!/_\theta T_B$ for the natural quotient map. Since $V /\!\!/_\theta T_B \subseteq W /\!\!/_\theta T_B$, and since the fibers of p are closed T_B-orbits, we obtain $V^s_\theta \subseteq W^s_\theta$. This implies that W^s_θ contains the torus T_C, because every point of $T_C \subset V$ is θ-stable by the first paragraph above. Then the irreducible component W of the scheme Z contains points of $Z \cap (\Bbbk^*)^{nr}$, which contradicts Theorem 3.10 since $W \neq V$. Thus, Y_θ is an irreducible component of $\mathcal{M}_\theta$. □

Remark 4.4. The adjective *coherent* comes from the theory of the toric Hilbert scheme introduced by Peeva–Stillman [PS02].

Remark 4.5. For finite abelian $G \subset \mathrm{SL}(n, \Bbbk)$ and $\theta \in \Theta$ abelian with $n \leq 3$, the projective birational morphism $\tau_\theta \colon Y_\theta \to \mathbb{A}^n_{\Bbbk}/G$ from Theorem 4.3 is a crepant resolution by Kronheimer [Kro89] and Bridgeland–King–Reid [BKR01]. Their results hold without the abelian assumption.

Theorem 4.3 suggests the following conjecture that does not require the abelian assumption on G. For any finite subgroup $G \subset \mathrm{GL}(n, \Bbbk)$, moduli spaces of θ-stable quiver representations are constructed as GIT quotients $\mathcal{M}_\theta = Z /\!\!/_\theta H$, where Z is an affine scheme, H is an algebraic group and $\theta \in H^* \otimes \mathbb{Q}$ is a fractional character (see Craw–Ishii [CI04, §2]). The G-orbit of the point $(1, \dots, 1) \in \mathbb{A}^n_{\Bbbk}$ defines a quiver representation $z \in Z$ that gives a point $[z] \in \mathcal{M}_\theta$ for all $\theta \in \Theta$. The algebraic torus $(\Bbbk^*)^n$ also acts on Z.

Conjecture 4.6. *Let V denote the subscheme of Z obtained as the closure of the $(\Bbbk^*)^n \times H$-orbit of $z \in Z$. For generic $\theta \in H^* \otimes \mathbb{Q}$, the GIT quotient $Y_\theta := V /\!\!/_\theta H$ is a reduced irreducible component of $\mathcal{M}_\theta$ that admits a projective birational morphism $Y_\theta \to Y_{\mathbf{0}} \cong \mathbb{A}^n_{\Bbbk}/G$ obtained by variation of GIT quotient.*

5. The G-Hilbert scheme

Applying Theorem 4.3 to the special case where $\mathcal{M}_\theta \cong G\text{-Hilb}$ provides a simple construction of Nakamura's G-Hilbert scheme. Assume that $G \subset \mathrm{GL}(n, \Bbbk)$ is abelian.

We first recall the G-Hilbert scheme. The literature contains two inequivalent definitions as follows. The first, denoted $G\text{-Hilb}$, is the fine moduli space of ideals $J \subseteq S$ defining G-invariant subschemes $Z(J) \subseteq \mathbb{A}^n_{\Bbbk}$ whose coordinate rings S/J are isomorphic to $\Bbbk G$ as a $\Bbbk G$-module. The ideal J, or the scheme $Z(J)$, is called a *G-cluster*, and we write $[J] \in G\text{-Hilb}$.

To define the second (and original) version, denoted Hilb^G, observe that the G-orbit of any point $p \in T^n \subseteq \mathbb{A}^n_{\Bbbk}$ consists of r distinct points permuted transitively by G. These orbits define points $[G \cdot p]$ in the G-fixed locus $(\mathrm{Hilb}^r(\mathbb{A}^n_{\Bbbk}))^G$ in the Hilbert scheme of r points in $\mathbb{A}^n_{\Bbbk}$. Every such point lies in a unique irreducible component of $(\mathrm{Hilb}^r(\mathbb{A}^n_{\Bbbk}))^G$ that we denote Hilb^G. To see this, note that an infinitesimal G-equivariant deformation of $[G \cdot p] \in \mathrm{Hilb}^r(\mathbb{A}^n_{\Bbbk})$ with $p \in T^n$ deforms the r distinct points in T^n that support the orbit. The resulting subscheme $Z' \subseteq \mathbb{A}^n_{\Bbbk}$ is supported on r distinct points in T^n and, since the deformation was G-equivariant, we obtain $Z' = G \cdot p'$ for some point $p' \in T^n$. This shows that G-clusters of the form $[G \cdot p]$ for $p \in T^n$ are open in some union of components of $(\mathrm{Hilb}^r(\mathbb{A}^n_{\Bbbk}))^G$. There is only one component since for any two such G-clusters $[G \cdot p], [G \cdot p'] \in G\text{-Hilb}$, one can construct a morphism $\mathbb{A}^1 \to G\text{-Hilb}$ whose image contains both $[G \cdot p]$ and $[G \cdot p']$. This shows that Hilb^G is well-defined.

Ito–Nakajima [IN00, §2] proved that $G\text{-Hilb}$ is a union of connected components of $(\mathrm{Hilb}^r(\mathbb{A}^n_{\Bbbk}))^G$. This also follows from Haiman–Sturmfels [HS04, Proposition 1.5]. The G-orbits $G \cdot p \subseteq \mathbb{A}^n_{\Bbbk}$ defined by points $p \in T^n$ are G-clusters, so the component of $(\mathrm{Hilb}^r(\mathbb{A}^n_{\Bbbk}))^G$ containing points of the form $[G \cdot p]$ for $p \in T^n$ is a component of $G\text{-Hilb}$. This is Hilb^G by definition. In particular, $\mathrm{Hilb}^G \subseteq G\text{-Hilb}$.

Remark 5.1. The original definition of Hilb^G is due to Ito–Nakamura [IN99] and further studied by Nakamura [Nak01]. The moduli definition $G\text{-Hilb}$ is due to Reid [Rei97] and is the version of the G-Hilbert scheme adopted by [IN00, BKR01, Ish02, HS04]. For a finite subgroup G of $\mathrm{GL}(2, \Bbbk)$ or $\mathrm{SL}(3, \Bbbk)$, it is known that $G\text{-Hilb}$ is smooth and connected (see [Ish02, BKR01]), hence $G\text{-Hilb} \cong \mathrm{Hilb}^G$.

These distinct versions of the G-Hilbert scheme can be constructed simultaneously via moduli of quiver representations as follows. Also, Nakamura [Nak01] asserted that Hilb^G is endowed with the reduced scheme structure, but here we show that this follows naturally from the definitions and Theorem 4.3.

Proposition 5.2. *Let $\theta \in \Theta$ satisfy $\theta_{\rho_0} < 0$ and $\theta_\rho > 0$ for $\rho \neq \rho_0$. Then there is an isomorphism $\mathcal{M}_\theta \cong G\text{-Hilb}$ that induces an isomorphism $Y_\theta \cong \mathrm{Hilb}^G$ by restriction. In particular, the scheme Hilb^G is reduced.*

Proof. Ito–Nakajima [IN00, §3] observed that there is a unique chamber $C_+ \subseteq \Theta$ in the GIT parameter space containing parameters $\{\theta \in \Theta \mid \theta_\rho > 0 \text{ for } \rho \neq \rho_0\}$ such that $\mathcal{M}_\theta \cong G\text{-Hilb}$ for all $\theta \in C_+$. To complete the proof of the first statement it remains to show that this isomorphism identifies the coherent component $Y_\theta \subseteq \mathcal{M}_\theta$ with $\mathrm{Hilb}^G \subseteq G\text{-Hilb}$. This follows from the proof of Theorem 4.3, where it is shown that the standard torus of Y_θ is isomorphic to the standard torus $T^n/G \subseteq \mathbb{A}^n_{\Bbbk}/G$ parametrizing G-orbits $G \cdot p$ for $p \in T^n$. The final statement follows since Y_θ is reduced by Theorem 4.3. □

Remark 5.3. The notation $\mathcal{M}_\theta \cong G\text{-Hilb}$ means that not only are the underlying schemes isomorphic but, in addition, the tautological bundles on both $\mathcal{M}_\theta$ and G-Hilb induced by the moduli constructions are also isomorphic as G-equivariant locally free sheaves.

Remark 5.4. Proposition 5.2 provides a direct GIT construction of the irreducible scheme $\mathrm{Hilb}^G \cong V /\!\!/_\theta T_B$ that avoids the Hilbert scheme of r-points in $\mathbb{A}^n_{\Bbbk}$, and shows that Hilb^G may be obtained from $\mathbb{A}^n/G$ by variation of GIT quotient. The Hilbert scheme of r-points in $\mathbb{A}^n_{\Bbbk}$ is in general much more singular than anything needed for Hilb^G.

6. Effective computation of the fan of Y_θ

The GIT construction of Y_θ allows an explicit description of the fan of the not-necessarily-normal toric variety Y_θ.

Definition 6.1. Let P_θ denote the convex hull of the set $\pi_n(\mathbb{N}C \cap \pi^{-1}(\theta))$ in the vector space $\pi_n(\pi^{-1}(\theta)) \otimes_{\mathbb{Z}} \mathbb{Q}$. Since the lattices $\pi^{-1}(\theta)$ and $\ker_{\mathbb{Z}}(\pi)$ are isomorphic, the proof of Proposition 4.1 gives $\pi_n(\pi^{-1}(\theta)) \cong M$. As a result, we regard P_θ as a polyhedron in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $\theta \in \Theta$.

Let F be a face of P_θ. The *inner normal cone* $\mathcal{N}_{P_\theta}(F)$ of P_θ at F is the set of all $\mathbf{y} \in M^\vee \otimes \mathbb{Q}$ such that the linear functional $\mathbf{y}$ is minimized over P_θ at F. The *inner normal fan* $\mathcal{N}(P_\theta)$ of P_θ is the polyhedral fan whose cones are $\{\mathcal{N}_{P_\theta}(F)\}$ as F varies over the faces of P_θ.

We now describe Y_θ as a not-necessarily-normal toric variety in terms of a fan (plus extra data) following Thompson [Tho03]. Replace the parameter θ by a positive multiple if necessary to ensure that the graded ring $\oplus_{j\geq 0}\Bbbk[V]_{j\theta}$ defining Y_θ is generated in degree one. Then Y_θ is covered by charts of the form $\mathrm{Spec}((\oplus_{j\geq 0}(\Bbbk[V]_{j\theta})_t)_0)$, where t varies over the generators of some ideal with radical the irrelevant ideal $\oplus_{j>0}\Bbbk[V]_{j\theta}$. We choose as a generating set the set of vertices of the convex lattice polyhedron P_θ. Let $\mathbf{m}$ be a vertex of P_θ, with $\sigma = \mathcal{N}_{P_\theta}(\mathbf{m})$, and let A_σ be the subsemigroup of M given by $A_\sigma := \mathbb{N}\langle \mathbf{p} - \mathbf{m} : \mathbf{p} \in P_\theta \cap M\rangle$.

Then the affine charts covering Y_θ are of the form $\operatorname{Spec}\Bbbk[A_\sigma]$, where σ varies over the normal cones of the vertices of P_θ.

If Y_θ is normal then the semigroups A_σ can be written as $A_\sigma = \sigma^\vee \cap M$. It follows that Y_θ is the toric variety with fan $\mathcal{N}(P_\theta) \subseteq M^\vee \otimes \mathbb{Q}$. Otherwise, $A_\sigma \subsetneq \sigma^\vee \cap M$ in general and Y_θ is described by a fan $(\Delta, \mathcal{S})$ in the sense of Thompson [Tho03] as follows. Consider the set $\Delta := \{\sigma \in \mathcal{N}(P_\theta)\}$ as the topological space whose open sets are the subfans of $\mathcal{N}(P_\theta)$, where each cone $\sigma \in \mathcal{N}(P_\theta)$ is regarded as the fan consisting of the faces of σ. We define a sheaf $\mathcal{S}$ of semigroups on Δ by first setting $\Gamma(\sigma, \mathcal{S}|_\sigma) := A_\sigma$ for $\sigma = \mathcal{N}_{P_\theta}(m)$. More generally, if τ is a face of $\sigma = \mathcal{N}_{P_\theta}(m)$, then $\Gamma(\tau, \mathcal{S}|_\tau) := A_\tau$ where $A_\tau = A_\sigma + \mathbb{Z}\mathbf{u}$ for $\tau = \sigma \cap \mathbf{u}^\perp$ (compare [Ful93, Lemma 1.3]). Then the pair $(\Delta, \mathcal{S})$ defines the nonnormal toric variety Y_θ as in Thompson [Tho03]. The point is simply that Y_θ is built from the local charts $\operatorname{Spec}\Bbbk[A_\sigma]$ rather than the normal varieties $\operatorname{Spec}\Bbbk[\sigma^\vee \cap M]$. In particular, the normalization of Y_θ is obtained by replacing each A_σ by its normalization $\sigma^\vee \cap M$.

Corollary 6.2. *The normalization $\widetilde{Y}_\theta$ of Y_θ is the toric variety whose fan is $\mathcal{N}(P_\theta)$. Furthermore, the fan $\mathcal{N}(P_\theta)$ is supported on the cone $(\mathbb{Q}^n_{\geq 0})^\vee$.*

Proof. It remains to prove the second statement. The morphism from Theorem 4.3 induces a projective, birational toric morphism $\widetilde{Y}_\theta \to \mathbb{A}^n_\Bbbk/G$, so the support of $\mathcal{N}(P_\theta)$ equals the support of the cone $(\mathbb{Q}^n_{\geq 0})^\vee$ in $M^\vee \otimes \mathbb{Q}$ defining $\mathbb{A}^n_\Bbbk/G$. ☐

We call $\mathcal{N}(P_\theta)$ the *fan of* Y_θ.

Remark 6.3. Sardo Infirri [SI96, Theorem 5.5] claimed that $\mathcal{M}_\theta$ is the toric variety with fan $\mathcal{N}(P_\theta)$ for all $\theta \in \Theta$. We show in [CMT05, Examples 4.12, 5.6] that $\mathcal{M}_\theta$ is not a normal toric variety in general.

Example 6.4. Let $G \cong \mathbb{Z}/r\mathbb{Z} \subset \mathrm{GL}(n, \Bbbk)$ be generated by $\operatorname{diag}(\omega, \dots, \omega)$, where ω is a primitive rth root of unity, so $\mathbb{A}^n_\Bbbk/G$ is of type $\frac{1}{r}(1, \dots, 1)$. The lattice $M^\vee$ is generated by the standard basis vectors plus the vector $\frac{1}{r}(1, \dots, 1)$. As a toric variety, $\mathbb{A}^n_\Bbbk/G$ corresponds to the rational cone generated by the standard basis vectors in $M^\vee \otimes \mathbb{Q}$. It is well known that a resolution $Y \to \mathbb{A}^n_\Bbbk/G$ is obtained by adding the ray generated by $\frac{1}{r}(1, \dots, 1) \in M^\vee$ and taking the stellar subdivision. This resolution is crepant if and only if $n = r$.

We now show that Y is isomorphic as a variety to Y_θ for all $\theta \in \Theta \smallsetminus \{0\}$. For $\theta \in \Theta$, the polyhedron $P_\theta = \pi_n\{C\mathbf{u} : B\mathbf{u} = \theta, \mathbf{u} \geq \mathbf{0}\}$. Since $\pi_n(C\mathbf{u}) = \sum u_i^\rho \mathbf{e}_i$, the minimum 1-norm of a vector in P_θ is $d_\theta := \min\{\sum u_i^\rho : \mathbf{u} \geq \mathbf{0}, B\mathbf{u} = \theta\}$, so $P_\theta \subseteq \{\mathbf{y} \in \mathbb{Q}^n_{\geq 0} : \sum_i y_i \geq d_\theta\}$. Since B is a unimodular matrix, we know that $d_\theta \in \mathbb{N}$. We claim that this inclusion is equality. First, we show that the vertices of the right hand side lie in P_θ. Indeed, by the unimodularity of B there exists $\mathbf{y} \in P_\theta \cap M$ whose 1-norm is d_θ. By replacing each $\mathbf{e}_i^\rho$ by $\mathbf{e}_1^\rho$, any vector $\mathbf{u} = \sum u_i^\rho \mathbf{e}_i^\rho \geq \mathbf{0}$ satisfying $C\mathbf{u} = (\theta, \mathbf{y})$ determines a vector $\mathbf{u}' := \sum u_i^\rho \mathbf{e}_1^\rho$ of type $d_\theta \mathbf{e}_1^\rho$. Since the McKay quiver of G is a cycle with n arrows connecting adjacent vertices, we have $B\mathbf{u}' = B\mathbf{u}$ and hence $C\mathbf{u}' = (\theta, d_\theta \mathbf{e}_1)$. This shows that the vertex $d_\theta \mathbf{e}_1$ and, similarly, any vertex $d_\theta \mathbf{e}_i$, lies in P_θ. To show that any point on a facet of $\{\mathbf{y} \in \mathbb{Q}^n_{\geq 0} : \sum_i y_i \geq d_\theta\}$ other than $\{\mathbf{y} \in \mathbb{Q}^n_{\geq 0} : \sum_i y_i = d_\theta\}$ also lies in P_θ, note that if γ is a cycle in the McKay quiver of type $r\mathbf{e}_i$ then $C(\mathbf{u}' + j\mathbf{v}(\gamma)) = (\theta, (d_\theta + jr)\mathbf{e}_i)$

for all $j > 0$. This proves that $P_\theta = \{\mathbf{y} \in \mathbb{Q}^n_{\geq 0} : \sum_i y_i \geq d_\theta\}$. Note that $d_\theta = 0$ if and only if $\theta = 0$. When $d_\theta > 0$, the normal fan of P_θ is the fan of the resolution $Y \to \mathbb{A}^n_{\Bbbk}/G$ defined above. In addition, it is easy to see that the semigroups satisfy $A_\sigma = \sigma^\vee \cap M$ for all top-dimensional cones σ in $\mathcal{N}(P_\theta)$. Thus Y_θ is normal, so $Y \cong Y_\theta$ for any $\theta \in \Theta \smallsetminus \{0\}$. It follows from Theorem 4.3 that for $\theta \in \Theta$ generic, the toric resolution Y of $\mathbb{A}^n_{\Bbbk}/G$ is isomorphic to the coherent component of the moduli space $\mathcal{M}_\theta$.

To emphasize that the previous results are explicit, we now give an algorithm to compute the fan of Y_θ. Every polyhedron P can be written as the Minkowski sum of a polytope Q and a polyhedral cone K. By the *generator representation* of P we mean the pair of lists L_1, L_2, where L_1 consists of the vertices of Q, and L_2 consists of the generators of K. The software package PORTA [Chr] converts between the inequality and generator representation of a polyhedron. Recall that the top $r \times nr$ submatrix of the matrix C is the vertex-edge incidence matrix B of the McKay quiver. Let D be the bottom $n \times nr$ submatrix of C.

Algorithm 6.5. To compute the fan of Y_θ.

Input: The GIT parameter $\theta \in \Theta$ and the matrix C.

1. Compute a generator representation for the polyhedron $\{\mathbf{u} \in \mathbb{Q}^{nr}_{\geq 0} : B\mathbf{u} = \theta\}$ and obtain the lists L_1 and L_2.
2. For $i = 1, 2$, replace the list L_i with the list $DL_i := \{D\mathbf{u} : \mathbf{u} \in L_i\}$.
3. Compute the inequality description of the polyhedron P_θ obtained as the sum of the polytope $\operatorname{conv}(DL_1)$ and the cone generated by DL_2.
4. The output of PORTA contains a table of incidences between inequalities of P_θ and vertices of P_θ. The normal fan at a vertex is generated by the negatives of normal vectors on the left hand side of those inequalities that hold at equality at the vertex.

Output: The inner normal fan of P_θ as a collection of sets of generators of normal cones at the vertices of P_θ.

Proof of Correctness. The fan of Y_θ is the inner normal fan of the lattice polyhedron $P_\theta = \operatorname{conv}(\pi_n(\mathbb{N}C \cap \pi^{-1}(\theta)))$. The set of lattice points $\mathbb{N}C \cap \pi^{-1}(\theta) = C(\{\mathbf{u} \in \mathbb{N}^{nr} : B\mathbf{u} = \theta\})$ is the image of $\{\mathbf{u} \in \mathbb{N}^{nr} : B\mathbf{u} = \theta\}$ under the linear map $C\colon \mathbb{Q}^{nr} \to \mathbb{Q}^{r+n}$. Further, $\pi_n(\mathbb{N}C \cap \pi^{-1}(\theta)) = D(\{\mathbf{u} \in \mathbb{N}^{nr} : B\mathbf{u} = \theta\})$ and hence $P_\theta = D(\operatorname{conv}(\{\mathbf{u} \in \mathbb{N}^{nr} : B\mathbf{u} = \theta\}))$ since the operation of taking convex hulls commutes with linear maps. Since the matrix B is totally unimodular (see Schrijver [Sch86, p274, Example 2]), we have $\operatorname{conv}(\{\mathbf{u} \in \mathbb{N}^{nr} : B\mathbf{u} = \theta\}) = \{\mathbf{u} \in \mathbb{Q}^{nr}_{\geq 0} : B\mathbf{u} = \theta\}$. Hence $P_\theta = D(\{\mathbf{u} \in \mathbb{Q}^{nr}_{\geq 0} : B\mathbf{u} = \theta\})$. This justifies steps (1) and (2) of the algorithm. Step (4) extracts the normal fan of P_θ by computing the normal cone at each vertex of P_θ. □

Example 6.6. Consider the action of type $\frac{1}{11}(1, 2, 8)$, so $G \subset \mathrm{SL}(3, \Bbbk)$ is the cyclic group of order 11 with generator $\operatorname{diag}(\omega, \omega^2, \omega^8)$ where ω is a primitive 11th root of unity. We compute the fan $\mathcal{N}(P_\theta)$ for $\theta = (1, 1, 1, 1, -7, -9, 1, 1, 1, 8, 1)$.

In this example, $r = 11$ and $n = 3$. The matrix C is a 14×33 matrix, and the polyhedron $\{\mathbf{u} \in \mathbb{Q}^{nr}_{\geq 0} : B\mathbf{u} = \theta\}$ is the Minkowski sum of a cone generated by 630 vectors and a polytope with 17581 vertices. After multiplying these lists of vectors by D, we compute $P_\theta \subset \mathbb{Q}^3$ as the sum of the convex hull of DL_1 and the cone generated by DL_2. Then P_θ is the sum of the cone generated by $(0,0,1), (0,1,0), (1,0,0)$ and the convex hull of $(0,0,78)$, $(0,21,15)$, $(0,26,11)$, $(0,70,0)$, $(22,0,23)$, $(96,0,0)$, $(4,0,50)$, $(4,9,23)$, $(4,46,0)$, $(72,0,3)$, $(4,34,3)$. Equivalently, P_θ is described by the irredundant inequalities listed in Table 1(a).

$$\begin{array}{llll}
(1) & -2x_1 - 4x_2 - 5x_3 & \leq & -159 \\
(2) & -3x_1 - 6x_2 - 2x_3 & \leq & -112 \\
(3) & -x_1 - 2x_2 - 8x_3 & \leq & -96 \\
(4) & -7x_1 - 3x_2 - x_3 & \leq & -78 \\
(5) & -6x_1 - x_2 - 4x_3 & \leq & -70 \\
(6) & -x_1 & \leq & 0 \\
(7) & -x_2 & \leq & 0 \\
(8) & -x_3 & \leq & 0
\end{array}$$

(a)

$(0,0,78)$	$\{4,6,7\}$
$(0,21,15)$	$\{1,4,6\}$
$(0,26,11)$	$\{1,5,6\}$
$(0,70,0)$	$\{5,6,8\}$
$(22,0,23)$	$\{1,2,7\}$
$(96,0,0)$	$\{3,7,8\}$
$(4,0,50)$	$\{2,4,7\}$
$(4,9,23)$	$\{1,2,4\}$
$(4,46,0)$	$\{3,5,8\}$
$(72,0,3)$	$\{1,3,7\}$
$(4,34,3)$	$\{1,3,5\}$

(b)

Table 1. (a) inequalities; (b) output from PORTA

To obtain the normal cones of P_θ we calculate the inequalities that hold at equality at which vertex. This information is carried in the strong validity table from PORTA at the end of the computation. We condense this information in Table 1(b). The first line of this table means that the normal cone at the vertex $(0,0,78)$ of P_θ is generated by the negatives of the coefficient vectors of the linear forms on the left hand side of inequalities (4), (6) and (7), in this case $(7,3,1), (1,0,0)$ and $(0,1,0)$. A cross-section of the fan $\mathcal{N}(P_\theta)$ is shown in Figure 4. The rays of $\mathcal{N}(P_\theta)$ are labeled 1 through 8 according to Table 1(a) so, for example, ray 6 is generated by $(1,0,0)$ and ray 3 is generated by $\frac{1}{11}(1,2,8)$.

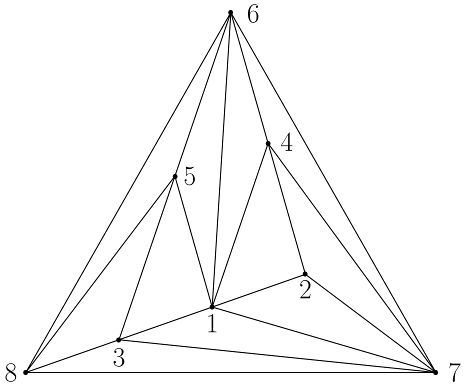

Figure 4. The fan of Y_θ for $\theta = (1,1,1,1,-7,-9,1,1,1,8,1)$

We note that while we have shown all steps in the algorithm explicitly in this example, this procedure can be completely automated.

Remark 6.7. The calculation of the toric fan defining Y_θ as in Figure 4 was originally done by hand by Craw [Cra01, §5.8.2] using a lengthy and somewhat speculative method. The method implemented here generalizes easily to calculate the fan for significantly more involved examples, and is automated.

7. Distinguished McKay quiver representations

In this section we calculate the distinguished θ-semistable quiver representations that define points on Y_θ. For generic $\theta \in \Theta$, these representations encode the restriction to Y_θ of the universal quiver representation on the fine moduli space $\mathcal{M}_\theta$ (see Craw–Ishii [CI04, §2]).

Recall from Theorem 3.10 that $V = \operatorname{Spec} \Bbbk[\mathbb{N}C]$, where C is the matrix with columns $C_i^\rho = \mathbf{e}_\rho - \mathbf{e}_{\rho\rho_i} + \mathbf{e}_i$. The standard torus of V is T_C, and, as for normal toric varieties, the orbits of the T_C-action correspond one-to-one to faces of the rational polyhedral cone $P := \mathbb{Q}_{\geq 0}C \subseteq \mathbb{Q}^{r+n}$ generated by the vectors C_i^ρ (see Sturmfels [Stu97, Proposition 1.3]). Specifically, the orbit corresponding to a face F of P is the intersection of $V \subseteq \mathbb{A}_\Bbbk^{nr}$ with the subscheme $\{(b_i^\rho) \in \mathbb{A}_\Bbbk^{nr} : b_i^\rho \neq 0 \text{ for } C_i^\rho \in F, b_i^\rho = 0 \text{ otherwise}\}$, and the *distinguished point* of this orbit $(b_i^\rho) \in V$ satisfies $b_i^\rho = 1$ for $C_i^\rho \in F$ and $b_i^\rho = 0$ otherwise. Note that the T_C-orbit of this distinguished point is the orbit associated with F. In particular, the standard torus $T_C \subseteq V$ corresponds to the full face P, and is the T_C-orbit of the distinguished point $(1, \dots, 1) \in \mathbb{A}_\Bbbk^{nr}$.

For $\theta \in \Theta$, the torus orbits of Y_θ correspond one-to-one to cones of $\mathcal{N}(P_\theta)$ or, equivalently, to faces of the polyhedron P_θ. Since $P_\theta = \pi_n(P \cap \pi^{-1}(\theta))$, every face of P_θ is of the form $F_\theta = \pi_n(\widetilde{F_\theta} \cap \pi^{-1}(\theta))$ where $\widetilde{F_\theta}$ is the smallest face of P containing the preimage of F_θ under the map π_n. The torus orbits of Y_θ correspond one-to-one to the θ-semistable T_C-orbits in V, and distinguished points of Y_θ correspond one-to-one to θ-semistable distinguished points of V.

Recall from Corollary 6.2 that the support of the fan $\mathcal{N}(P_\theta)$ is $(\mathbb{Q}^n_{\geq 0})^\vee$.

Definition 7.1. For $\theta \in \Theta$ and $\mathbf{w} \in (\mathbb{Q}^n_{\geq 0})^\vee$, the *distinguished θ-semistable representation* $b_{\theta,\mathbf{w}} = (b_i^\rho)$ is the distinguished point of V corresponding to the cone of $\mathcal{N}(P_\theta)$ containing $\mathbf{w}$ in its relative interior.

Computing these representations has been the key tool in understanding the moduli space $\mathcal{M}_\theta$ (see [Nak01, Rei97, Cra01]). No reasonable algorithm was known to compute these representations for $\mathcal{M}_\theta \neq G$-Hilb, and the algorithm for G-Hilb introduced by Nakamura required that one perform a sequence of *G-igsaw transformations* to calculate a single G-cluster. These transformations are the G-Hilb analogues of *flips* for the toric Hilbert scheme introduced by Maclagan–Thomas [MT02].

The next theorem presents an elementary method to compute any distinguished θ-semistable quiver representation in one step. Note that the cone dual to P is $P^\vee = \{(\mathbf{v}, \mathbf{w}) \in (\mathbb{Q}^r)^* \times (\mathbb{Q}^n)^* : w_i + v_\rho - v_{\rho\rho_i} \geq 0\}$. Given $\mathbf{w} \in (\mathbb{Q}^n_{\geq 0})^\vee$, consider the slice $P^\vee_{\mathbf{w}} := \{\mathbf{v} \in (\mathbb{Q}^r)^* : w_i + v_\rho - v_{\rho\rho_i} \geq 0\}$.

Theorem 7.2. *Fix $\theta \in \Theta$ and $\mathbf{w} \in (\mathbb{Q}^n_{\geq 0})^\vee$. Let $\mathbf{v} \in P^\vee_{\mathbf{w}}$ be any vector with $\theta \cdot \mathbf{v} \leq \theta \cdot \mathbf{v}'$ for all $\mathbf{v}' \in P^\vee_{\mathbf{w}}$. The distinguished θ-semistable quiver representation $b_{\theta,\mathbf{w}} = (b^\rho_i)$ has*

$$b^\rho_i = \begin{cases} 1 & \textit{if } w_i + v_\rho - v_{\rho\rho_i} = 0 \\ 0 & \textit{if } w_i + v_\rho - v_{\rho\rho_i} > 0 \end{cases} . \tag{7.1}$$

Proof. Write $\mathrm{face}_{\mathbf{w}}(P_\theta)$ for the face of P_θ where $\mathbf{w}$ is minimized, and $F := \widetilde{\mathrm{face}_{\mathbf{w}}}(P_\theta)$ for the smallest face of P containing the preimage of $\mathrm{face}_{\mathbf{w}}(P_\theta)$ under the map π_n. The distinguished point $b_{\theta,\mathbf{w}} \in V$ satisfies $b^\rho_i = 1$ if $C^\rho_i \in F$ and $b^\rho_i = 0$ otherwise. To restate this condition in terms of weight vectors, note that $\mathbf{v} \in P^\vee_{\mathbf{w}}$ implies $(\mathbf{v}, \mathbf{w}) \in P^\vee$, so $w_i + v_\rho - v_{\rho\rho_i} \geq 0$ for all $1 \leq i \leq n$ and $\rho \in G^*$. Furthermore, $(\mathbf{v}, \mathbf{w})$ lies in the face $\mathcal{N}_P(F)$ of $P^\vee$ if and only if $w_i + v_\rho - v_{\rho\rho_i} = 0$ for $C^\rho_i \in F$ and $w_i + v_\rho - v_{\rho\rho_i} > 0$ otherwise. In particular, the distinguished point $b_{\theta,\mathbf{w}} \in V$ satisfies the conditions of (7.1) if and only if $(\mathbf{v}, \mathbf{w})$ lies in the face $\mathcal{N}_P(F)$.

We complete the proof by showing that for $\mathbf{v}$ satisfying the hypothesis of the theorem, the vector $(\mathbf{v}, \mathbf{w})$ lies in the cone $\mathcal{N}_P(F)$. The hypothesis states that $\mathbf{v}$ lies in $\mathrm{face}_\theta(P^\vee_{\mathbf{w}})$, the face of $P^\vee_{\mathbf{w}}$ where θ is minimized. Thus, $(\mathbf{v}, \mathbf{w})$ lies in the smallest face of $P^\vee$ containing $\mathrm{face}_\theta(P^\vee_{\mathbf{w}})$. The fact that $\mathcal{N}_P(F)$ is the smallest face of $P^\vee$ containing $\mathrm{face}_\theta(P^\vee_{\mathbf{w}})$ is the content of Craw–Maclagan [CM05, Proposition 2.7]. Thus $(\mathbf{v}, \mathbf{w}) \in \mathcal{N}_P(F)$. □

Remark 7.3. To compute the quiver representation $b \in V$ corresponding to a point $[b] \in Y_\theta$ that is not distinguished, first compute the distinguished point $b_{\theta,\mathbf{w}}$ in the same T_C-orbit as $b \in V$, and then let T_C act on the coordinates of $b_{\theta,\mathbf{w}}$.

Example 7.4. For $\mathbf{w} = \mathbf{0} \in (\mathbb{Q}^n_{\geq 0})^\vee$, we obtain the inequalities $v_\rho \geq v_{\rho\rho_i}$ for all $\rho \in G^*$, $1 \leq i \leq n$. Since every arrow of the McKay quiver lies in some directed cycle, these inequalities must be equalities, so the quiver representation $b_{\theta,\mathbf{0}}$ satisfies $b^\rho_i = 1$ for all $1 \leq i \leq n$ and $\rho \in G^*$.

Remark 7.5. When $\mathbf{w} \in (\mathbb{Q}^n_{\geq 0})^\vee$ is a point of the lattice $M^\vee$, the inequalities defining $P^\vee_{\mathbf{w}}$ form the *reductor condition* of Logvinenko [Log03, Equation (4.8)]. Thus, Algorithm 7.6 enables Logvinenko to verify whether the G-constellations arising from his reductor sets are θ-stable for any given θ.

Theorem 7.2 gives an explicit algorithm to compute the distinguished McKay quiver representation $b_{\theta,\mathbf{w}}$ for $(\theta, \mathbf{w}) \in \Theta \times (\mathbb{Q}^n_{\geq 0})^\vee$, which we now present.

Algorithm 7.6. To compute the distinguished representation $b_{\theta,\mathbf{w}}$.

Input: $(\theta, \mathbf{w}) \in \Theta \times (\mathbb{Q}^n_{\geq 0})^\vee$ and the matrix C.

1. Compute the polyhedron $P^\vee_{\mathbf{w}} = \{\mathbf{v}' \in \mathbb{Q}^r : (\mathbf{v}', \mathbf{w})C \geq \mathbf{0}\}$.
2. Compute an optimal solution $\mathbf{v}$ of the linear program
$$\mathrm{minimize}\{\theta \cdot \mathbf{v}' : \mathbf{v}' \in P^\vee_{\mathbf{w}}\}.$$
3. The distinguished quiver representation $b_{\theta,\mathbf{w}} = (b^\rho_i)$ has coordinates
$$b^\rho_i = \begin{cases} 1 & \text{if } w_i + v_\rho - v_{\rho\rho_i} = 0 \\ 0 & \text{if } w_i + v_\rho - v_{\rho\rho_i} > 0 \end{cases} .$$

Proof of Correctness. This is immediate from Theorem 7.2. □

Example 7.7. For the group action of type $\frac{1}{11}(1,2,8)$ considered in Example 6.6, the given three-dimensional representation decomposes as $\rho_1 \oplus \rho_2 \oplus \rho_8$. We now compute a pair of θ-stable distinguished quiver representations for the parameter $\theta = (1,1,1,1,-7,-9,1,1,1,8,1)$ considered in Example 6.6.

The vector $\mathbf{w} = (10,7,6)$ lies in the relative interior of the cone generated by $(2,4,5)$, $(7,3,1)$ and $(1,0,0)$ corresponding to vertices 1, 4 and 6 in Figure 4. The vector $\mathbf{v} = (-8,-10,-1,-3,6,4,-9,0,-2,-15,-6)$ is an optimal solution to the linear program in Step (2) from either algorithm from Algorithm 7.6. Calculating the vector $(\mathbf{v},\mathbf{w})C = (12,0,0,1,0,11,12,0,11,1,0,11,12,22,22,23,11,11,1,0,0,12,22,0,23,11,0,1,0,0,12,11,0)$, shows that the distinguished representation is

$$b_{\theta,\mathbf{w}} = (0,1,1,0,1,0,0,1,0,0,1,0,0,0,0,0,0,0,0,1,1,0,0,1,0,0,1,0,1,1,0,0,1),$$

where the coordinates of $b_{\theta,\mathbf{w}} = (b_i^{\rho})$ are indexed exactly as the columns of the matrix C. Figure 5 shows only the arrows a_i^{ρ} for which $b_i^{\rho} \neq 0$.

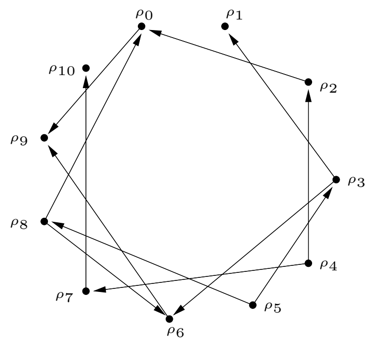

FIGURE 5. The subquiver consisting of arrows a_i^{ρ} for which $b_i^{\rho} \neq 0$

Example 7.8. For the group action from Example 7.7, consider the vector $\mathbf{w} = (8,3,1)$ in the relative interior of the two dimensional cone of $\mathcal{N}(P_\theta)$ generated by the vectors $(1,0,0)$ and $(7,3,1)$ corresponding to vertices 6 and 4 in Figure 4. In this case, the optimal solution is $\mathbf{v} = (-5,-9,-2,-6,1,-3,-7,0,-4,-8,-1)$. The product $(\mathbf{v},\mathbf{w})C$ has 18 entries equal to zero, and

$$b_{\theta,\mathbf{w}} = (0,1,1,0,1,1,0,1,1,0,1,1,0,0,0,0,1,1,0,1,1,0,0,1,0,1,1,0,1,1,0,0,1).$$

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