MATH 551 HOMEWORK 8

SOLUTIONS

(1) (a) Let $I = \{r_1a_1 + \cdots + r_na_n : r_i \in R, 1 \leq i \leq n\}$. If $x = \sum r_ia_i, y = \sum s_ja_j \in I$, then $x - y = \sum(r_i - s_i)a_i \in I$. If $x = \sum r_ia_i \in I$, then $rx = \sum(rr_i)a_i \in I$. Since R is commutative, $xr = rx \in I$. Thus I is an ideal. For each i, we set $r_i = 1_R$, and $r_j = 0_R$, to see that $a_i \in I$ for all i, so I is an ideal of R containing a_i for all i, and thus $\langle a_i : 1 \leq i \leq n\rangle \subseteq I$. However for all choices of r_i for $1 \leq i \leq n$, $r_ia_i \in \langle a_i : 1 \leq i \leq n\rangle$, and thus $\sum r_ia_i \in \langle a_i : 1 \leq i \leq n\rangle$. We thus conclude that $\langle a_i : 1 \leq i \leq n\rangle = I$.

(b) Let B be the left submodule generated by $a \in A$. Let $C = \{ra : r \in R\}$. Let $ra, sa \in C$, for $r, s \in R$. Then $ra - sa = (r - s)a \in C$, since $r - s \in R$, so C is a subgroup of A containing $1_Ra = a$. Also if $ra \in C$, then $s(ra) = (sr)a \in C$, since $sr \in R$. Thus C is a left submodule of A containing a, so $B \subseteq C$. Now if D is any left submodule of A containing a, then $ra \in D$ for $r \in R$, so $C \subseteq D$, and thus C is contained in B, and thus $C = B$.

(c) Let R be the ring of 2×2 matrices with real entries, and let

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then every element of $\{ra : r \in R\}$ is a matrix with the first column all zero, but we know that the ideal generated by a is all of R, since R has no proper nontrival ideals, so $\{ra : r \in R\}$ is not equal to the ideal generated by a. However we did not use commutativity in the second part of the question.

(2) Let $R = k[x]$ be the polynomial ring in one variable over a field k, and let $A = \{x^i : i \geq 0\} = B$. Then $x + 1 = (x)(1) + (1)(1)$, but $x + 1$ is not the product of two elements f and g for $f \in A$ and $g \in B$.

(3) Let M be a cyclic R module, generated by $a \in M$. Let $\phi : R \to M$ be the function given by $\phi(r) = ra$. This is an R-module homomorphism, since $\phi(r + s) = (r + s)a = ra + sa = \phi(r) + \phi(s)$,

and $\phi(rs) = rsa = r(sa) = r\phi(s)$. The homomorphism ϕ is surjective, by the first problem (since we didn't need commutativity) so $M \cong R/\ker(\phi)$. Now R-submodules of R are (left) ideals of R, so $\ker(\phi)$ is a left ideal of R.

(4) **Hungerford IV.1.5**

(a) Let M be a simple R-module, and let $a \neq 0 \in R$. Let N be the submodule generated by a. Since $a \neq 0$, we have $N \neq 0$, so since M is simple, we must have $N = M$.

(b) Let $\phi : A \to A$ be an R-module homomorphism. Now $\ker(\phi)$ is a submodule of A, so is either 0 or A. If $\ker(\phi) = A$, then ϕ is the zero map. If $\ker(\phi) = 0$, then ϕ is injective. Now $\phi(A)$ is a submodule of A, so if $A \neq 0$, since $\ker(\phi) = 0$ we must have $\phi(A) = A$, and thus ϕ is a bijective homomorphsim, and thus an isomorphism. If $A = 0$, then the only endomorphism is the zero map, so the claim is trivially true.

(5) **Hungerford IV.1.9** Let $0 \to \ker(f) \to A \to \text{im}(f) \to 0$ be a short exact sequence, with the first map the inclusion of $\ker(f)$ into A, and the second f. Now let $g : \text{im}(f) \to A$ be the R-module homomorphism that is the inclusion of $\text{im}(f)$ into A. Then $fg(f(a)) = f(f(a)) = f(a)$ for all $f(a) \in \text{im}(f)$, so $fg = 1_{\text{im}(f)}$, so the short exact sequence splits, and thus $A \cong \ker(f) \oplus \text{im}(f)$.

If R is a field, so A is a vector space, then f is a projection.

(6) Let R be the ring of p-quaternions, and let I be a nontrivial ideal. Let $r = a+bi+cj+dk \in I$. If $n(r) = a^2+b^2+c^2+d^2 \neq 0$, then $n(r)$ is invertible in $\mathbb{Z}/p\mathbb{Z}$, so $n(r)^{-1}(a-bi-cj-dk)r = 1_R$, so $I = R$. Thus if $I \neq R$ we must have $n(r) = 0$ for all $r \in R$. Let r be as above, and consider $ri + ir = -2b + 2ai$, $rj + jr = -2c + 2aj$, and $rk + kr = -2d + 2ak$. Since $n(f) = 0$ for these f if $I \neq R$, we have $4a^2+4b^2 = 4a^2+4c^2 = 4a^2+4d^2 = a^2 + b^2 + c^d + d^2 = 0$. Since $p > 2$, 4 is invertible in $\mathbb{Z}/p\mathbb{Z}$, so $a^2 + b^2 = a^2 + c^2 = a^2 + d^2 = 0$, so $0 = a^2 + b^2 + c^2 + d^2 = c^2 + d^2 = -2a^2$, so again since 2 is invertible in $\mathbb{Z}/p\mathbb{Z}$, $a^2 = 0$, and thus, since $\mathbb{Z}/p\mathbb{Z}$ has no zerodivisors, $a = 0$. But from this it follows that $b = c = d = 0$ as well, so $r = 0$. Thus if $I \neq R$, then $I = 0$, so R is a simple ring.