MATH 551 HOMEWORK 6

SOLUTIONS

(1) **Hungerford II.4.9** Suppose that $G/C(G)$ is cyclic, generated by $aC(G)$. Let $H = \langle a \rangle$ be the subgroup of G generated by a. Then $HC(G) = G$, since every left coset of $C(G)$ has the form $a^k C(G)$ for some k. Also, $H \cap C(G) = \{e\}$, and $C(G) \lhd G$. So (by the exam question!) $G \cong C(G) \rtimes H$, where the map $H \to \mathrm{Aut}(C(G))$ maps $h \mapsto \tau_h$, where $\tau_h(g) = hgh^{-1}$. Since $C(G)$ is the center of G, all τ_h are trivial, so $C(G) \rtimes H \cong C(G) \times H$. Thus since $C(G)$ and H are both cyclic groups, so is G.

(2) **Hungerford II.4.14.** Suppose that $|G| = pn$, with $p > n$ prime, and H is a subgroup of order p. By the Sylow theorems the number of Sylow p-subgroups is one mod p, and divides $|G|$, and thus divides n. Since $n < p$, there must be only one such subgroup, so H is normal.

(3) **Hungerford II.5.7** We start with S_3. The Sylow 2 subgroups have order 2, so there are three, each generated by the three transpositions. There is one (normal) Sylow 3 subgroup, generated by (123). Since $|S_4| = 24 = (2^3)3$, the Sylow 2 subgroups have order 8, while the Sylow 3 subgroups have order 3. There are four of the latter, generated by the elements $\{(123), (124), (134), (234)\}$. To see this, note that they must be cyclic, generated by an element of order three, and there are eight of these, half of which are the inverses of the other half. The Sylow 2-subgroups are all isomorphic to D_4. For example, we have $\{1, (1234), (13)(24), (1432), (13), (24), (14)(23), (12)(34)\}$. Since all Sylow 2 subgroups are conjugate, and conjugation by an element of S_n conjugates the base set, to list the other Sylow 2-subgroups it suffices to list the vertices of the square acted on by D_4 in clockwise order. The options are: $\{(1234), (1243), (1324)\}$.

Now $|S_5|$ has order $120 = (2^3)(3)(5)$, so the Sylow 2 subgroups again have order 8, while the Sylow 3 subgroups have order 3. There are $\binom{5}{3} = 10$ different Sylow 3 subgroups, similarly to above (generated by (ijk) for different choices of sets $\{i, j, k\} \subset \{1, 2, 3, 4, 5\}$. The Sylow 2 subgroups are again isomorphic to

D_4, with the 15 permuted squares being: $\{(1234), (1243), (1324),$ $(1235), (1253), (1325), (1254), (1245), (1524), (1534), (1543),$ $(1354), (5234), (5243), (5324)\}$.

(4) **Hungerford II.5.9** Suppose that $|G| = p^n q$, with $p > q$ prime. By the Sylow theorems, the number of Sylow p-subgroups is one mod p, and divides $|G|$, and thus divides q. Since $q < p$, there must be only one of them, so the unique Sylow p subgroups is thus normal. Since (again by the Sylow theorems) every subgroup of order p^n (index q) of G is a Sylow subgroup, G contains a unique subgroup of index q.

(5) **Hungerford II.5.13** By the class equation we know that the center of a group of order p^2 is nontrivial. If the center has size p^2, the group is automatically abelian, so we only need to consider the case where $|C(G)| = p$. But then $G/C(G)$ has order p, so is cyclic, so by the first question we know that G is abelian.