MATH 551 HOMEWORK 5

SOLUTIONS

1. **Hungerford II.2.11 a and b**
   - (a) Let $\{m_1, \dots, m_r\}$ be the invariant factors of G, so $G \cong \mathbb{Z}^s \oplus \bigoplus_{i=1}^{r} \mathbb{Z}/m_i\mathbb{Z}$. Let $\{n_1, \dots, n_k\}$ be the invariant factors of H, so $H \cong \mathbb{Z}^l \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/n_i\mathbb{Z}$. Now $G \oplus G \cong \mathbb{Z}^{2s} \oplus \bigoplus_{i=1}^{r} \mathbb{Z}/m_i\mathbb{Z} \oplus \mathbb{Z}/m_i\mathbb{Z}$, and $H \oplus H \cong \mathbb{Z}^{2l} \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/n_i\mathbb{Z} \oplus \mathbb{Z}/n_i\mathbb{Z}$, so the rank of the free part of $G \oplus G$ is $2s$, which must equal $2l$, so $s = l$. Also, the invariant factors of $G \oplus G \cong H \oplus H$ are $\{m_1, m_1, m_2, m_2, \dots, m_r, m_r\}$, or $\{n_1, n_1, n_2, n_2, \dots, n_k, n_k\}$. Since both lists are ordered in successively dividing order, they must be the invariant factors of $G \oplus G$, so $n = k$, and $n_i = m_i$ for each i. Thus G and H have the same rank and invariant factors, so are isomorphic.
   - (b) Let $A = \{p_1^{a_1}, \dots, p_r^{a_r}\}$ be the elementary divisors of H, while $B = \{q_1^{b_1}, \dots, q_s^{b_s}\}$ are the elementary divisors of K, and $C = \{l_1^{c_1}, \dots, l_t^{c_t}\}$ are the elementary divisors of G. Then the elementary divisors of $G \oplus H$ is the multiset $A \cup C$, while the elementary divisors of $G \oplus K$ is the multiset $B \cup C$. If $A \neq B$, we will have $A \cup C \neq B \cup C$, but $A \cup C = B \cup C$ as a multiset, so we must have $A = B$, so H and K have the same elementary divisors. If G has rank d while H has rank e and K has rank f, then $G \oplus H$ has rank $d + e$, while $G \oplus K$ has rank $d + f$, so $d + e = d + f$ implies that $e = f$, so H and K have the same elementary divisors, and rank, and so are isomorphic.
2. **Hungerford II.2.12**
   - (a) The elementary divisors of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/35\mathbb{Z}$ are $\{2, 3^2, 5, 7\}$, while the invariant factor is 630. For $\mathbb{Z}/26\mathbb{Z} \oplus \mathbb{Z}/42\mathbb{Z} \oplus \mathbb{Z}/49\mathbb{Z} \oplus \mathbb{Z}/200\mathbb{Z} \oplus \mathbb{Z}/1000\mathbb{Z}$ the elementary divisors are $\{2, 2, 2^3, 2^3, 3, 5^2, 5^3, 7, 7^2\}$, while the invariant factos are $\{(2^3)(3)(5^3)(7^2), (2^3)(5^2)(7), 2, 2\}$.
   - (b) There are 11 abelian groups of order 64, corresponding to the 11 partitions of 6: $6 = 5 + 1 = 4 + 2 = 4 + 1 + 1 = 3 + 3 = 3 + 2 + 1 = 3 + 1 + 1 + 1 = 2 + 2 + 2 = 2 + 2 + 1 + 1 =$

$2+1+1+1+1 = 1+1+1+1+1+1$. For example $3+1+1+1$ corresponds to the group $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. There are 7 abelian groups of order 96. They correspond to the partitions of 5: $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$. For example, $3+1+1$ corresponds to the group $\mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

(c) There is just the trivial group of order one. For each prime p there is only one group of order p, $\mathbb{Z}/\mathbb{Z}$. This deals with $n = 2, 3, 5, 7, 11, 13, 17, 19$. For $n = p^2$, p prime, the two abelian groups are $\mathbb{Z}/p^2\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. This deals with $n = 4, 9, 16$. For $n = pq$ for p, q both prime, the only abelian group is $\mathbb{Z}/pq\mathbb{Z}$, which deals with $n = 6, 10, 14, 15$. For $n = 8$ the three abelian groups are $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. For $n = p^2q$ with p, q prime, the two abelian groups are $\mathbb{Z}/p^2q\mathbb{Z}$ and $\mathbb{Z}/pq\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. This deals with $n = 12, 18, 20$, which are the last cases.

(3) **A *partition* of $n \in \mathbb{N}$ is a sequence $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_k$ where $\lambda_i \in \mathbb{N}$, $\lambda_i > 0$, and $\sum_{i=1}^{k} \lambda_i = n$. Show that the conjugacy classes of S_n correspond to partitions of n.**

Given a decomposition of $\pi \in S_n$ into disjoint cycles $\pi = C_1 \ldots C_r$, including cycles of length one, let λ_i be the length of C_i. Then $\sum_{i=1}^{r} \lambda_i = n$, and we may assume that the λ_i are nonincreasing, so we have a partition of n, called the *cycle type* of π.

Let $\pi, \tau \in S_n$. Then $\tau\pi\tau^{-1}(\tau(i)) = \tau(\pi(i))$, so if $\pi = (a_1 a_2 \ldots a_k)$ is a cycle, then $\tau\pi\tau^{-1} = (\tau(a_1)\tau(a_2)\ldots\tau(a_k))$. If we write a general $\pi \in S_n$ as a product of disjoint cycles $\pi = C_1 C_2 \ldots C_r$, then $\tau\pi\tau^{-1} = (\tau C_1 \tau^{-1})(\tau C_2 \tau^{-1})\ldots(\tau C_r \tau^{-1})$. Since τ is a bijection, this is a decomposition of $\tau\pi\tau^{-1}$ into disjoint cycles, so $\tau\pi\tau^{-1}$ has the same cycle type as π.

Conversely, if π and π' have the same cycle type, write $\pi = C_1 \ldots C_r$ and $\pi' = C'_1 \ldots C'_r$, where C_i and C'_i have the same length. For each i, choose $j \in C_i$, and $k \in C'_i$, and define $\tau \in S_n$ by $\tau(j) = k$, and $\tau(\pi^s(j)) = \pi'^s(k)$, for s less than the length of C_i. This defines a bijection from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$ since the domain is everything occuring in π, and the range is everything occuring in π'. Thus $\tau \in S_n$. By construction we have $\tau\pi\tau^{-1} = \pi'$, so π and π' are conjugate.

(4) **Hungerford II.4.3** Consider G acting on itself by conjugation. Then the size of the orbit of a is the index in G of the stabilizer of $a = C_G(a) = \{g \in G : gag^{-1} = a\}$, so $C_G(a)$ is an

index 2 subgroup of G, which is thus normal (and proper). If $C_G(a) = e$, then since it is index 2 we would have $|G| = 2$, and so $G \cong \mathbb{Z}/2\mathbb{Z}$ is abelian, and thus every conjugacy class has size one, contradicting our hypothesis on a. So $C_G(a)$ is the desired proper nontrivial normal subgroup.

(5) **Hungerford II.4.8** Since $\mathbb{Z}/6\mathbb{Z}$ is abelian, $gxg^{-1} = x$ for all $g, x \in \mathbb{Z}/6\mathbb{Z}$, so the only inner automorphism is the trivial one. Thus the other automorphism ϕ of $\mathbb{Z}/6\mathbb{Z}$, for which $\phi(1) = 5$ (so $\phi(x) = -x$ for all $x \in \mathbb{Z}/6\mathbb{Z}$) is not an inner automorphism.

(6) **How many different necklaces can be made with seven beads, if the beads are of three different colours?**

Consider the dihedral group D_7 acting on the regular 7-gon with each vertex coloured one of three possible colours. We want to know the number of orbits of this action. By "the lemma that is not Burnside's" this is $1/14 \sum_{g \in D_7} |X_g|$. For the identity $|X_g| = 3^7$. For any rotation $|X_g| = 3$, while for any reflection $|X_g| = 3^4$, so we have the number of orbits is $1/14(3^7 + 6(3) + 7(3^4)) = 198$. Thus there are 198 different necklaces.