MATH 551 HOMEWORK 4

SOLUTIONS

THESE SOLUTIONS ARE VASTLY INCOMPLETE, BUT ARE BEING POSTED EARLY AND WILL BE UPDATED WHEN I RETURN TO TOWN ON MONDAY.

(1) **Hungerford I.9.3** Let $y \in F$, and $x^n \in N$. Then $yx^ny^{-1} = (yxy^{-1})^n \in N$, so $yNy^{-1} \subseteq N$, and thus N is normal.

(2) **Hungerford I.9.8** The dihedral group D_n of symmetries of a regular polygon is generated by the counterclockwise rotation by $2\pi/n$ and the reflection in a line through two vertices, τ. We define a map ϕ from $F(a,b)$ to D_4 by sending a to σ and b to τ. Since $\sigma^n = 1$, $\tau^2 = 1$, and $\tau\sigma = \sigma^{-1}\tau$, we have $a^n, b^2, bab^{-1}a \in \ker(\phi)$, so there is a mab from $F(a,b)/R$ to D_4, where R is then normal subgroup generated by $a^n, b^2, andbab^{-1}a$. It remains to show that this map is an isomorphism. FINISH.

(3) **Hungerford II.1.2**
   (a) Suppose first that X is linearly independent. Let $f = n_1x_1 + \dots n_kx_k \in \langle X \rangle$, where $x_i \in X$, and $n_i \in \mathbb{Z}$. Suppose also that $f = m_1x_1 + \dots m_kx_k$. (Note that in principle this other description of f may involve different elements of X, but by enlarging k and setting appropriate coefficients to 0 we may assume that the same set $x_1, \dots, x_k$ appears in both descriptions.) Then $0 = f - f = (n_1 - m_1)x_1 + \cdots + (n_k - m_k)x_k$. The assumption that X is linearly independent now implies that $n_i = m_i$ for all i. Conversely, suppose that such descriptions are unique for all elements of $\langle X \rangle$. Since $0 \in \langle X \rangle$, if $n_1x_1 + \dots n_kx_k = 0$, we must therefore have $n_i = 0$ for all i.
   (b) Let $F = \mathbb{Z}$. Then F has the basis $\{1\}$, so has rank 1. However the set $\{2\}$ is linearly independent (since it has infinite order) but is not a basis (since $1 \neq n2$ for $n \in \mathbb{Z}$).
   (c) Let $F = \mathbb{Z}$. Suppose that the linearly independent set $\{2\}$ is contained in some basis B, and let $n \in B$. Then $n2 + (-2)n = 0$, but $n, -2 \neq 0$, so B is not linearly independent, a contradiction.

(d) Let $F = \mathbb{Z}$. Then $\{2, 3\}$ generates $\mathbb{Z}$ (since 1 generates $\mathbb{Z}$, and $1 = 3 + (-1)2$), and neither of $\{2\}$ or $\{3\}$ is a basis for $\mathbb{Z}$. ???rank bound.

(4) **Hungerford II.1.10**

(a) Suppose that $\{q_1, \ldots, q_k\}$ generate $\mathbb{Q}$. Write $q_i = a_i/b_i$ where $(a_i, b_i) = 1$ and $b_i > 0$. Let m be the least common multiple of the b_i. Then any element of the form $n_1 q_1 + \ldots n_k q_k$ can be written in the form s/m where $s \in \mathbb{Z}$. Then $1/(m+1)$ cannot be written in the form s/m with $s \in \mathbb{Z}$, as if it could we would have $m = s(m+1)$, but $|s(m+1)| > m$, so this is impossible. Thus $\mathbb{Q}$ is not finitely generated.

(b) Suppose that $\mathbb{Q}$ were free, so it had a basis X. By the first part we know that X is infinite. Let $a/b, c/d \in X$. Then $bc(a/b) + -ad(c/d) = 0$, but $bc, -ad \neq 0$, so X is not linearly independent, a contradiction.

(c) FINISH.

(5) **Hungerford II.1.11** Let p_i be the ith prime number. Let $f = \sum_{i=0}^{k} a_i x^k \in G$, where $a_i \in \mathbb{Z}$. Define a map $\phi : G \to \mathbb{Q}_{>0}$ by $\phi(f) = \prod_{i=0}^{k} p_{i+1}^{a_i}$. Note that ???

(6) *Show that the group of automorphisms of $\mathbb{Z}^n$ is isomorphic to the group of $n \times n$ matrices with integer entries and determinant ± 1.*

Note that since $\det(AB) = \det(A)\det(B)$, the set of $n \times n$ integer matrices with determinant ± 1 is closed under multiplication. Since $\det(A)\det(A^{-1}) = \det(I) = 1$, we see that the inverse of such a matrix also has determinant ± 1. ???CHECK INTEGER??

Fix the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_i$ of $\mathbb{Z}^n$, where $\mathbf{e}_i$ has a one in the ith position, and 0 elsewhere. Note that a homomorphism of $\mathbb{Z}^n$ is determined by where the $\mathbf{e}_i$ are sent, and all choices of n elements of $\mathbb{Z}^n$ determine a homomorphism from $\mathbb{Z}^n$ to $\mathbb{Z}^n$. Fix an automorphism ϕ, and write $\phi(\mathbf{e}_j) = \sum_{i=1}^{n} a_{ij}\mathbf{e}_i$, where $a_{ij} \in \mathbb{Z}$. Form the $n \times n$ matrix A_ϕ with $(i, j)th$ entry a_{ij}. This gives a map from automorphisms to $n \times n$ matrices, $\phi \mapsto A_\phi$. FINISH.

(7) **Written Qualifying Exam, Fall 2004**: *Let G be the set of all complex 3×3 matrices which have exactly one nonzero element in every row and in every column. Show that G is a group under matrix multiplication. Show that G has two normal subgroups G_1 and G_2 with $G_1 \subset G_2 \subset G$ such that G_1, G_2/G_1 and G/G_2*

are all abelian groups. (The original question let you assume that G is a group).

FIRST SHOW THAT G IS A GROUP.

Let G_1 be the 3×3 diagonal matrices with all diagonal entries nonzero. Since the product and inverse of diagonal matrices are diagonal, and the identity matrix is invertible, this is a subgroup of G. CHECK THAT THIS IS NORMAL.

Let G_2 be the matrices A in G with the property that if $A_{ii} \neq 0$ for some $1 \leq i \leq 3$, then A is diagonal.

FINISH.