# MATH 551 HOMEWORK 3

SOLUTIONS

(1) **Hungerford I.7.1** If $(S, x)$, $(S', x')$ are two pointed sets, then the set of functions from $S$ to $S'$ that take $x$ to $x'$ is a subset of the set of all functions from $S$ to $S'$, so each $\hom((S, x), (S', x'))$ is a set. The composition of two morphisms $f : (S, x) \to (S', x')$, and $g : (S', x') \to (S'', x'')$ is the composition of functions $gf : (S, x) \to (S'', x'')$. Since $f(x) = x'$, and $g(x') = x''$, the function $gf$ is a morphism in this category, so we can compose morphisms. Composition is associative because composition of functions is associative. The identity function $(S, x) \to (S, x)$ is the identity element of $\hom((S, x), (S, x))$. Thus pointed sets form a category.

(2) **Hungerford I.7.2** Suppose that $f : A \to B$ is an equivalence in $\mathcal{C}$, and $g, h$ are two morphisms from $B$ to $A$ with $g \circ f = 1_A = h \circ f$, and $f \circ g = 1_B = f \circ h$. Then $(g \circ f) \circ h = 1_A \circ h = h$, and $g \circ (f \circ h) = g \circ 1_B = g$, so $g = h$.

(3) **Hungerford X.1.1a** Let $F_O$ be the function that takes a group $G$ to the set of all its subgroups. Let $F_M$ be the function that takes a homomorphism $\phi$ from $G$ to $H$, and returns the function $\phi_S$ from the set of subgroups of $G$ to the set of subgroups of $H$ given by $\phi_S(G') = \phi(G')$. Note that this is well-defined, since $\phi(G')$ is the image of a homomorphismm from $G'$ into $H$, so is a subgroup of $H$. The definition of composition and its associativity follow from the associativity of homomorphisms between groups. If $\phi : G \to H$, and $\phi' : H \to H'$, then $\phi' \circ \phi(G') = \phi'(\phi(G))$, so $F_M(\phi' \circ \phi) = F_M(\phi') \circ F_M(\phi)$. If $\phi$ is the identity homomorphism, then $\phi_S$ is the identity maps from subgroups of $G$ to subgroups of $G$, and so $F_M(1_G) = 1_{F_O(G)}$, and so $F$ is a functor.

(4) **Hungerford X.1.2**
- Let $\mathcal{C}$ be the category with four objects $A, B, C, D$, two non-identity morphisms: $f : A \to B$, and $g : C \to D$, and the four identity morphisms. This is a category, as for any pair of composable morphisms one must be the identity, so composition works as expected.

Let $\mathcal{D}$ be the category with three objects $a, b, d$, and three non-identity morphisms, $h : a \to b$, $j : b \to d$, and $k : a \to d$, with $j \circ h = k$. Let $T$ be the functor with $T(A) = a$, $T(B) = b = T(C)$, $T(D) = d$. The map on morphisms is defined by sending the identity morphisms to the corresponding identity morphisms, and $T(f) = h$, $T(g) = j$. Note that $h, j \in \mathrm{im}(T)$ but $j \circ h = k \notin \mathrm{im}(T)$, so $\mathrm{im}(T)$ is not a category, as composition of morphisms is not defined.

- We have identity morphisms in $\mathrm{im}(T)$ by the functor axiom $T(1_C) = 1_{T(C)}$, and associativity of composition follows from associativity in $\mathcal{D}$, so we just need to show that the composition is defined. If $A, B, C \in \mathrm{im}(T)$, let $A = T(a)$, $B = T(b)$, and $C = T(c)$ for $a, b, c \in \mathcal{C}$. If $F \in \hom(A, B)$, with $F \in \mathrm{im}(T)$, then $F = T(f)$ for $f \in \hom(a', b')$, where $T(a') = A$, $T(b') = B$. Since $T$ is injective, we must have $a' = a$, and $b' = b$, so $f \in \hom(a, b)$. Similarly, if $G \in \hom(B, C)$, then $G = T(g)$ for $g \in \hom(b, c)$. Thus $G \circ F = T(g) \circ T(f) = T(g \circ f) \in \mathrm{im}\, T$, so composition of morphisms is defined, and so $\mathrm{im}(T)$ is a category.

(5) **Hungerford X.1.3** The map $TS$ on objects takes $C \in \mathcal{C}$ to $T(S(T))$, and on morphisms takes $f$ to $T(S(f))$. Then $TS(1_C) = T(S(1_C)) = T(1_{S(C)}) = 1_{TS(C)}$. To show that $TS$ is a functor we need to check that it behaves correctly with composition. Let $f : A \to B$, and $g : B \to C$.

If $\sigma(T) = \sigma(S) = 1$, then $S(f) : S(A) \to S(B)$, and $T(S(f)) : T(S(A)) \to T(S(B))$, so $TS(f) : TS(A) \to TS(B)$. Also $TS(g \circ f) = T(S(g \circ f)) = T(S(g) \circ S(f)) = TS(g) \circ TS(f)$, so $TS$ is a covariant functor, so $\sigma(TS) = 1 = (1)(1)$.

If $\sigma(T) = \sigma(S) = -1$, then $S(f) : S(B) \to S(A)$, and $T(S(f)) : T(S(A)) \to T(S(B))$, so $TS(f) : TS(A) \to TS(B)$. Also $TS(g \circ f) = T(S(g \circ f)) = T(S(f) \circ S(g)) = TS(g) \circ TS(f)$, so $TS$ is a covariant functor, so $\sigma(TS) = 1 = (-1)(-1)$.

If $\sigma(T) = 1, \sigma(S) = -1$, then $S(f) : S(B) \to S(A)$, and $T(S(f)) : T(S(B)) \to T(S(A))$, so $TS(f) : TS(B) \to TS(A)$. Also $TS(g \circ f) = T(S(g \circ f)) = T(S(f) \circ S(g)) = TS(f) \circ TS(g)$, so $TS$ is a contravariant functor, so $\sigma(TS) = -1 = (-1)(1)$.

If $\sigma(T) = -1, \sigma(S) = 1$, then $S(f) : S(A) \to S(B)$, and $T(S(f)) : T(S(B)) \to T(S(A))$, so $TS(f) : TS(B) \to TS(A)$. Also $TS(g \circ f) = T(S(g \circ f)) = T(S(g) \circ S(f)) = TS(f) \circ TS(g)$, so $TS$ is a contravariant functor, so $\sigma(TS) = -1 = (1)(-1)$.

(6) **Hungerford X.1.4a** Let $\mathcal{C}$ be the subcategory of the category of sets whose objects are product $A \times B$ of sets. The morphisms in $\mathcal{C}$ are functions (so morphisms in the category of sets). Let $1 : \mathcal{C} \to \mathcal{C}$ be the identity functor. Let $F : \mathcal{C} \to \mathcal{C}$ be the functor that takes $A \times B$ to $B \times A$. If $f : A \times B \to C \times D$ is a morphism in $\mathcal{C}$, so $f((a,b)) = (f_1(a,b), f_2(a,b))$, then $F(f) : B \times A \to D \times C$ is defined by $f((b,a)) = (f_2(a,b), f_1(a,b))$. We claim that $F$ and 1 are naturally isomorphic. We define $\alpha : 1 \to F$ by setting $\alpha_{A\times B} : 1(A \times B) \to F(A \times B)$ to be given by $\alpha_{A\times B}((a,b)) = (b,a)$. Note that $\alpha_{A\times B}$ is a morphism in $\mathcal{C}$. Then the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{\alpha_{A\times B}} & B \times A \\ {\scriptstyle 1(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ C \times D & \xrightarrow[\alpha_{C\times D}]{} & D \times C \end{array}$$

as $F(f) \circ \alpha_{A\times B}((a,b)) = F(f)(b,a) = (f_2(a,b), f_1(a,b)) = \alpha_{C\times D} \circ f$. Thus $\alpha$ is a natural transformation. Since each map $\alpha_{A\times B}$ is an equivalence, this map is a natural isomorphism.

To show that there is a natural bijection $(A \times B) \times C \to A\times(B\times C)$, we let $\mathcal{D}$ be the category whose objects are cartesian products of three sets, and whose morphisms are inherited from the category of sets. We then have two functors $S$ and $T$ from $\mathcal{D}$ to $\mathcal{C}$: $S(A \times B \times C) = (A \times B) \times C$, and $T(A \times B \times C) = A\times(B\times C)$. The relevant $\alpha_{A\times B\times C} : (A\times B)\times C \to A\times(B\times C)$ is given by $\alpha_{A\times B\times C}(((a,b),c)) = (a,(b,c))$, which is a morphism in $\mathcal{C}$. Check that $S$ and $T$ are functors, and that the appropriate diagram commutes.

(7) **Let $X$ be a topological space, and let $\mathcal{X}$ be the category whose objects are the open sets of $X$, and for which there is a morphism $U \to V$ exactly when $U \subseteq V$, in which case $\hom(U,V)$ is a single element. Do products always exist in this category?**

Yes. Let $\{A_i : i \in I\}$ be a family of open sets in $X$, and let $P$ be the interior of $\cap_{i\in I} A_i$. Then $P$ is an open set contained in all $A_i$, so there is a morphism $\pi_i$ from $P$ to $A_i$ for each $i$. If $B$ is an open set of $X$ with a morphism $f_i : B \to A_i$, then $B \subseteq A_i$ for all $i$, so $B \subseteq \cap_i A_i$, and thus, since $B$ is open, $B \subseteq P$, so there a morphism $\phi : B \to P$. Since both $f_i$ and $\pi_i \circ \phi$ are morphisms from $B$ to $A_i$, and there is only one such morphism, they must be the same, $f_i = \pi_i \circ \phi$. Similarly, the morphism $\phi$ is unique, so $P$ satisfies the product axioms.

(8) **Let $F$ be the map from $\mathcal{X}$ to the category of groups that takes an object $U$ of $\mathcal{X}$ to the group of continuous functions from $U$ to $\mathbb{R}$. Show that you can define a map on morphisms in such a way that $F$ becomes a functor. (This is an example of *presheaf*).**

Write $C(U)$ for the group of continuous functions from $U$ to $\mathbb{R}$. Given a morphism $f : U \to V$, we must have $U \subseteq V$, so set $F(f)$ to the be map from $C(V)$ to $C(U)$ defined by restriction to $U$, so for $\phi \in C(v)$, $F(f)(\phi) = \phi|_U$. Note that $\phi_U$ is a continuous function, since $\phi$ is. Also note that $F(f)(\phi + \psi) = F(f)(\phi) + F(f)(\psi)$, so $F(f)$ is a group homomorphism.

If $f = 1_U$ for some $U$, then $F(f)$ is the identity map from $C(U)$ to $C(U)$, since $\phi|_U = \phi$, so $F(1_U) = 1_{F(U)}$. If $f : U \to V$ and $g : V \to W$, then $F(f) \circ F(g) : C(W) \to C(U)$ sends $\phi \in C(W)$ to $(\phi|_V)_U = \phi_U$. Since $g \circ f$ is the inclusion of $U$ into $W$, this is also $F(g \circ f)$, so $F(g \circ f) = F(f) \circ F(g)$, so we conclude that $F$ is a contravariant functor from $\mathcal{X}$ to the category of groups.