MATH 551 HOMEWORK 1

SOLUTIONS

(1) **Hungerford, I.1.3. (p 29)** No. A counter-example is given by the following multiplication table:

	e	x
e	e	x
x	e	x

Note that this multiplication is associative, and e is a left identity element. The element e is the right inverse of both e and x. This is not a group, however (for example, there is no right identity element).

(2) **Hungerford, I.1.13. (p 30)** Fix $a, b \in G$. Then $(ab)^2 = abab = e$, so $ab = b^{-1}a^{-1}e = ba$, where the first equality comes from multiplying on the left by $b^{-1}a^{-1}$, and the second from the realization that since $b^2 = a^2 = e$, $b = b^{-1}$ and $a = a^{-1}$. Thus G is abelian.

(3) **Hungerford, I.1.14. (p 30)** Consider the decomposition $G = \cup_{a\in G}\{a, a^{-1}\}$. These sets have size either one or two, with size one exactly if $a = a^{-1}$. Now $e = e^{-1}$, so there is at least one set of size one, so since $|G|$ is even there must be another $a \neq e$ with $a = a^{-1}$, as otherwise we would have an even $|G|$ equal to $1 + 2k$, where k is the number of $\{a, a^{-1}\}$ pairs of size two. But this means there is $a \neq e$ with $a^2 = e$.

(4) **Hungerford, I.2.3. (p 33)** As in the hint, observe (by multiplying matrices) that $BA = A^3B$, and $A^4 = B^4 = I$. The first observation means that we can write any element of Q_8 as A^jB^k for $0 \leq j, k \leq 3$. Also note that $A^2 = B^2$, so we may assume that $k \in \{0, 1\}$. This shows that Q_8 has size at most 8. We can write out the corresponding matrices to check that it has size exactly 8. Since $BA = A^3B \neq AB$, Q_8 is not abelian.

(5) **Hungerford, I.2.4. (p 33)** First check that $C^4 = D^2 = I$, and $DC = C^3D$, so we can write all elements of H in the form C^jD^k where $0 \leq j \leq 4$ and $k \in \{0, 1\}$. This shows that $|H| \leq 8$. Multiplying out the matrices we can check that $|H| = 8$. To see that H is isomorphic to D_4, let ϕ be the function that sends C^jD^k to $\sigma^j\tau^k$, where σ is the rotation by 90 degrees

counterclockwise, and τ is a horizontal reflection. Checking the homomorphism property reduces to noting that $\tau\sigma = \sigma^3\tau$. The map ϕ is a bijection, and thus also an isomorphism.