MATH 551 HOMEWORK 11

SOLUTIONS

(1) **Hungerford, IV.5.2**

(a) Define a map $\phi : A \to A \otimes \mathbb{Z}/m\mathbb{Z}$ by $\phi(a) = a \otimes 1$. Now $\phi(ma) = ma \otimes 1 = a \otimes m = a \otimes 0 = 0$, so $mA \in \ker \phi$, and so there is map $\overline{\phi} : A/mA \to A \otimes \mathbb{Z}/m\mathbb{Z}$. We also define a map $\psi : A \otimes \mathbb{Z}/m\mathbb{Z} \to A/mA$ by $\psi(a \otimes j) = ja$ (Check that this map is well-defined!). Note that $\psi(\phi(a)) = a$, while $\phi(\psi(a \otimes j)) = ja \otimes 1 = a \otimes j$, so ϕ and ψ are inverses, and thus each is an isomorphism.

(b) By the first part, $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong (\mathbb{Z}/m\mathbb{Z})/(n\mathbb{Z}/m\mathbb{Z})$. Let $c = (m, n)$, and let $\phi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/c\mathbb{Z}$ be the map that takes $x + m\mathbb{Z}$ to $x + c\mathbb{Z}$. This is well-defined, since if $x - x' \in m\mathbb{Z}$, then $x - x' \in c\mathbb{Z}$, since c divides m, and is a homomorphism. The homomorphism ϕ is surjective, and $\phi(nx) = nx + c\mathbb{Z} = 0$, since c divides n, so $n\mathbb{Z}/m\mathbb{Z} \subseteq \ker(\phi)$. If $x + m\mathbb{Z} \in \ker(\phi)$, then c divides x. Since $c = (m, n)$, there are $r, s \in \mathbb{Z}$ with $rm+sn = c$, so $x = r'm+s'n$, and thus $x + m\mathbb{Z} = s'n + m\mathbb{Z}$, so $x \in n(\mathbb{Z}/m\mathbb{Z})$. Thus $\ker(\phi) = n(\mathbb{Z}/m\mathbb{Z})$, and so $Z/c\mathbb{Z} \cong (\mathbb{Z}/m\mathbb{Z})/(n\mathbb{Z}/m\mathbb{Z})$ by the isomorphism theorems.

(c) By the classification of finitely generated abelian groups, we know that $A \cong \mathbb{Z}^a \oplus \oplus_i \mathbb{Z}/d_i\mathbb{Z}$, where d_i divides d_{i+1}. Similarly $B \cong \mathbb{Z}^b \oplus \oplus_j \mathbb{Z}/e_j\mathbb{Z}$, where e_i divides e_{i+1}. Then

$$\begin{aligned} A \otimes B &= (\mathbb{Z}^a \oplus \bigoplus_i \mathbb{Z}/d_i\mathbb{Z}) \otimes (\mathbb{Z}^b \oplus \bigoplus_j \mathbb{Z}/e_j\mathbb{Z}) \\ &= \mathbb{Z}^{a+b} \oplus \bigoplus_i \mathbb{Z}/d_i\mathbb{Z} \oplus \bigoplus_j \mathbb{Z}/e_j\mathbb{Z} \oplus \bigoplus_{i,j} \mathbb{Z}/(d_i, e_j)\mathbb{Z}. \end{aligned}$$

(2) **Hungerford, IV.5.4**

(a) Let $\mathbb{Z}[i]$ be the group $\{a+bi : a, b \in \mathbb{Z}\}$, with multiplication $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$. As a group $\mathbb{Z}[i]$ is a free abelian of rank two. Now $\mathbb{Z}[i] \otimes_{\mathbb{Z}[i]} \mathbb{Z}[i] \cong \mathbb{Z}[i]$, by Theorem 5.7, so is free abelian of rank two. However $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Z}[i]$ is free abelian of rank four, with basis $1 \otimes 1$,

$1 \otimes i$, $i \otimes 1$, and $i \otimes 1$. Note that we can replace i here by $\sqrt{d}$ for any non-perfect-square d.
Warning: Many people choose examples where one of their groups did not have an R-module structure!

(b) Let V be a two-dimensional vector space over a field k, with basis $\mathbf{e}_1, \mathbf{e}_2$, and consider $V \otimes_k V$. We show below that this is a vector space over k with basis $\mathbf{e}_i \otimes \mathbf{e}_j$ for $1 \leq i, j \leq 2$. Then $a \otimes b = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2) \otimes (b_1\mathbf{e}_1 + b_2\mathbf{e}_2) = a_1b_1\mathbf{e}_1 \otimes \mathbf{e}_1 + a_1b_2\mathbf{e}_1 \otimes \mathbf{e}_2 + a_2b_1\mathbf{e}_2 \otimes \mathbf{e}_1 + a_2b_2\mathbf{e}_2 \otimes \mathbf{e}_2$. Now consider $u = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2$. If $u = a \otimes b$, then $a_1b_1 = 1$, and $a_2b_2 = 1$, so none of a_1, a_2, b_1, b_2 are zero. But then $a_1b_2, a_2b_1 \neq 0$, so u should have some $\mathbf{e}_1 \otimes \mathbf{e}_2$ and $\mathbf{e}_2 \otimes \mathbf{e}_1$ components. Thus we conclude that $u \neq a \otimes b$ for any $a, b \in V$.

(c) In the previous question we showed that $\mathbb{Z}/3\mathbb{Z} \otimes \mathbb{Z}/5\mathbb{Z} = 0$, so $1 \otimes 1 = 2 \otimes 2$, but $1 \neq 2$.

(3) **Hungerford, IV.5.7** By an earlier question we know that $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \neq 0$, and $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \neq 0$. To show that $1 \otimes \alpha$ is the zero map, we only need to show that $1 \otimes \alpha(1 \otimes 1) = 0$, since $1 \otimes 1$ is the only nonzero element of $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z}$. Indeed, $1 \otimes \alpha(1 \otimes 1) = 1 \otimes \alpha(1) = 1 \otimes 2 = 2 \otimes 1 = 0 \otimes 1 = 0$, so $1 \otimes \alpha$ is the zero map.

(4) **Hungerford, IV.5.9**

(a) Let ϕ be the group homomorphism $\phi : R/I \otimes_R B \to B + IB$ given by setting $\phi((r+I) \otimes b) = rb + IB$. (This really means defining the map on the free group on $R/I \times B$ and noting that the tensor relations are in the kernel, so we have a map from $R/I \otimes_R B$.) To see that this map is well-defined, not that if $r' + I = r + I$, then $r'b - rb = (r - r')b \in IB$, since $r - r' \in I$. Since R has 1_R, $\phi(1 \otimes b) = b + IB$, so ϕ is surjective. Finally, if $\sum_i (r_i + I) \otimes b_i \in \ker \phi$, then $\sum_i r_i b_i \in IB$. Then $\sum_i (r_i + I) \otimes b_i = \sum_i ((1_R + I) \otimes r_i b_i) = (1_R + I) \otimes (\sum_i r_i b_i) = (1_R + I) \otimes 0 = 0$, so ϕ is injective and thus an isomorphism.

(b) By the first part we have $R/I \otimes_R R/J \cong (R/I)/J(R/I)$, so we need to show that this latter module is isomorphic to $R/(I + J)$. Indeed, consider the map $\phi : R/J \to R/(I + J)$ given by $\phi(r + J) = r + I + J$. Check that this is a well-defined R-module homomorphism. Note that $J(R/I) \in \ker(\phi)$. If $r + I \in \ker(\phi)$, then $r \in I + J$, so we can write $r = i + j$ for $i \in I$, $j \in J$. Then $r + I = j + I$, so $r + I \in J(R/I)$, so $\ker(\phi) = J(R/I)$, and thus $(R/I)/J(R/I)$ is isomorphic as an R-module to $R/(I+J)$ by the isomorphism theorems.

(5) *If A is the matrix of a linear map ϕ from an n-dimensional vector space V over a field k to itself, and B is the matrix of a linear map ψ from an m-dimensional vector space W over k to itself, what is the matrix of $\phi \otimes \psi : V \otimes_k W \to V \otimes W$?*

Let the basis of V with respect to which A is the matrix be $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, and the basis of W for B be $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$. Then a basis for $V \otimes_k W$ is $\{\mathbf{e}_i \otimes \mathbf{f}_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. This follows from the fact that tensor product commutes with direct sum, and that $k \otimes_k k \cong k$.

Then $\phi \otimes \psi(\mathbf{e}_i \otimes \mathbf{f}_j) = \phi(\mathbf{e}_i) \otimes \psi(\mathbf{f}_j) = \sum_{k=1}^{n} \sum_{l=1}^{m} a_{ki} b_{lj} \mathbf{e}_k \otimes \mathbf{e}_l$, so the corresponding matrix has $A \otimes B_{(k,l),(i,j)} = a_{ki} b_{lj}$. Order the basis so that $\mathbf{e}_i \otimes \mathbf{f}_j$ comes before $\mathbf{e}_k \otimes \mathbf{f}_l$ if $i < k$, or $i = k$ and $j < l$. Then matrix is:

$$\left(\begin{array}{c|c|c|c} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ \hline a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{n1}B & a_{n2}B & \ldots & a_{nn}B \end{array}\right).$$

(6) *Show that the map from the category of R-modules to itself that takes an R module A to $A \otimes_R B$, where B is a fixed R-module, and takes a morphism f to $f \otimes 1$ is a functor.*

Denote this pair of maps by T. We need to show that $T(1_C) = 1_{T(C)}$ for all objects C in the category, and that $T(g \circ f) = T(g) \circ T(f)$ for any two morphisms f, g in the category whose composition is defined.

For the first, if C is an R-module, the function $1_C \otimes 1 : C \otimes B \to C \times B$ takes $c \otimes b$ to $1_C(c) \otimes 1(b) = c \otimes b$, so is the identity map.

For the second, if $f : C \to D$ and $g : D \to E$ are R-module homomorphisms, then $T(f) : C \otimes B \to D \otimes B$ is given by $T(f)(c \otimes b) = f(c) \otimes b$, and $T(g) : D \otimes B \to E \otimes B$ is given by $T(g)(d \otimes b) = g(d) \otimes b$, so $T(g) \circ T(f)(c \otimes b) = g(f(c)) \otimes b$, so $T(g) \circ T(f) = T(g \circ f)$, as required, so T is a covariant functor.

(7) *Compute the Jordan and rational canonical forms of the following matrix:*

$$\begin{pmatrix} 1 & 3 & 0 \\ -1 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix}.$$

The characteristic polynomial of the given matrix A is $(x-1)^3$, so the only eigenvalue is 1. The eigenspace corresponding to 1 is one-dimensional, being the span of $(3,0,1)$. Note that at this point we can already answer the question: there must be only one Jordan block of size three, so the minimal polynomial must also be $(x-1)^3$, and thus the Jordan and rational canonical forms of A are:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{pmatrix}.$$

To compute this directly, we compute the space E_1^2, which is two-dimensional, spanned by $(3,0,1)$ and $(0,1,0)$. Finally, E_1^3 is the whole space. So a basis for E_1^3/E_1^2 is $v=(1,0,0)$. We then have $(A-I)v=(0,-1,0)$, and $(A-I)^2v=(-3,0,-1)$. So in the ordered basis $\{(-3,0,-1),(0,-1,0),(1,0,0)\}$ the matrix for A is in Jordan canonical form. The matrix for A with respect to the ordered basis $\{(1,0,0),(1,-1,0),(-2,-2,-1)\}$ is in rational canonical form.

(8) **Hungerford VII.4.5** See Jordan canonical form handout.

(9) **Hungerford VII.4.13** It suffices to show that a matrix in Jordan canonical form is similar to its transpose, as then if $J=CAC^{-1}$, and $J^t=DJD^{-1}$, then $A^t=(C^{-1}JC)^t=C^tJ^t(C^{-1})^t=C^tDJD^{-1}(C^{-1})^t=C^tDCAC^{-1}D^{-1}(C^{-1})^t=(C^tDC)A(C^tDC)^{-1}$. To show the claim about a Jordan matrix, it suffices to show it for a single Jordan block. But if

$$J=\begin{pmatrix} \lambda & 1 & \dots & 0 & 0 \\ 0 & \lambda & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix},$$

then J^t is the matrix for the same linear transformation with the same basis listed in the opposite order.

(10) **Fall 2001** *Suppose that V is a real vector space of finite dimension n and $T: V \to V$ is a linear transformation with no repeated eigenvalues. Show that there exists a vector $v \in V$ such that $\{v, Tv, T^2v, \dots, T^{n-1}v\}$ is a basis of V.*

If T has no repeated eigenvalues, then the minimal polynomial equals the characteristic polynomial, so the rational canonical form consists of just one block, being the companion matrix of the minimal polynomial. For a companion matrix the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ has the required form.

(11) **Fall 2002** *Suppose that A and B are $n \times n$ complex nilpotent matrices with the same rank and the same minimal polynomial.*

(a) *If $n = 6$, prove that A and B are similar.*
Since A and B are nilpotent with the same minimal polynomial, that polynomial is x^k for some k, and all the eigenvalues are zero. Thus the Jordan canonical form consists of a number of Jordan blocks with eigenvalue zero. There is one zero columns for each Jordan block, and each other column has exactly one one, which is the only one in its row. Thus the rank of the matrices A and B is six minus the number of Jordan blocks, while the k from the minimal polynomial gives the size of the largest Jordan block. Since A and B have the same rank and minimal polynomial, they have the same number of Jordan blocks, and the same size of the largest Jordan block. Jordan forms of 6×6 nilpotent matrices correspond to partitions of 6, so the options are: $6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1$. Each of these is determined by fixing the largest number and the number of parts, so A and B must have the same Jordan canonical form, and are thus similar.

(b) *If $n = 7$, is this still true? Either prove or give a counterexample.* This is not true if $n = 7$. An example is given by the partitions $3+3+1 = 3+2+2$ of 7, corresponding to the matrices:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

both of which have rank four and minimal polynomial x^3.

(12) **Spring 2004** *Let $f(x) = (x-1)(x+1)^2$ and $g(x) = (x-1)(x+1)(x+2)$ in $\mathbb{Q}[x]$. find a 3×3 rational matrix A such that $g(A) = 0$, and the characteristic polynomial of A is $-f$. Here the characteristic polynomial is defined as $\det(A - xI)$.*

We know that the minimal polynomial must divide both f and g, and that A must have eigenvalues 1 and -1, with the dimension of the generalized eigenspace of -1 being two. Thus the minimal polynomial must be $(x-1)(x+1)$. This says that the size of the largest Jordan block is one for both eigenvalues. Thus the Jordan canonical form (which is an example of such a matrix) must be:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$