1 Preliminarics

1.1 A review of linear algebra

**Vector Space** Let \( \mathbb{R} \) be the scalar field of real numbers. We consider only real vector spaces. Let \( V_n \) be a set. \( V_n \) is a vector space (also called a linear space) if it is equipped with two operations:

- **scalar product** \( \mathbb{R} \times V_n \to V_n \),
- **vector addition** \( V_n \times V_n \to V_n \),

and it is closed under these two operations. That is, \( V_n \) is a vector space if \( \forall \alpha, \beta \in \mathbb{R} \) and \( \forall a, b \in V_n \),

\[ \alpha a + \beta b \in V_n. \]

The vector space \( V_n \) is \( n \)-dimensional if we can find a basis \( \{ e_1, \ldots, e_n \} \subset V_n \) such that for any \( a \in V_n \), we have a unique decomposition

\[ a = \sum_{i=1}^{n} a_i e_i, \]

where \( a_i \in \mathbb{R} \) (\( i = 1, \ldots, n \)) are the components (coordinates) of vector \( a \) under the basis \( \{ e_1, \ldots, e_n \} \).

**Tensor Space** Let \( V_n \) (\( V_m \)) be \( n \)-dimensional (\( m \)-dimensional) vector space. A mapping \( A : V_n \to V_m \) is a tensor if \( A \) is linear. That is, \( \forall \alpha, \beta \in \mathbb{R} \) and \( \forall a, b \in V_n \),

\[ A(\alpha a + \beta b) = \alpha A(a) + \beta A(b). \quad (1) \]

Let \( \text{Lin}(V_n, V_m) \) be the collection of all linear mappings (i.e., tensors) with domain \( V_n \) and range \( V_m \). For any \( \alpha \in \mathbb{R} \) and any \( A_1, A_2 \in \text{Lin}(V_n, V_m) \), define two operations

- **scalar product** \( (\alpha A_1)(a) = \alpha A_1(a) \quad \forall a \in V_n \),
- **vector addition** \( (A_1 + A_2)(a) = A_1(a) + A_2(a) \quad \forall a \in V_n \).

\[ \blacklozenge \text{Claim:} \text{ For any } \alpha, \beta \in \mathbb{R} \text{ and any } A_1, A_2 \in \text{Lin}(V_n, V_m), \alpha A_1 + \beta A_2 \text{ is a linear mapping (from } V_n \text{ to } V_m). \]

The above claim implies that the set \( \text{Lin}(V_n, V_m) \) is also a vector space.

**Inner Product** We equip a \( n \)-dimensional vector space \( V_n \) with a mapping \( V_n \times V_n \to \mathbb{R} \), called inner product such that for any \( \alpha, \beta \in \mathbb{R} \) and any \( a, b, c \in V_n \), the inner product is

1. **Positive-definite**: \( a \cdot a \geq 0; \ a \cdot a = 0 \iff a = 0 \),
2. **Linear**: \( a \cdot (\alpha b + \beta c) = \alpha a \cdot b + \beta a \cdot c \),
3. **Symmetric**: \( a \cdot b = b \cdot a \).

Geometric interpretations:

- **Length of a vector**: \( |a| = \sqrt{a \cdot a} \),
- **Angle between two vectors**: \( \cos(\theta) = \frac{a \cdot b}{|a||b|} \).
Euclidean Space $\mathbb{R}^n$. For a $n$-dimensional vector space $V_n$ equipped with an inner product, we can find an orthonormal basis $\{e_i : i = 1, \cdots, n\}$ such that for all $i, j = 1, \cdots, n$,

$$e_i \cdot e_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where $\delta_{ij}$ is called kronecker delta. With respect to this basis, for any vector $a \in V_n$, we find its components $(a_1, \cdots, a_n)$ (or coordinates if $a$ is a point in space)

$$a = \sum_{i=1}^{n} a_i e_i, \quad a_i = a \cdot e_i \in \mathbb{R} \quad \forall i = 1, \cdots, n.$$ 

We can further identify the space $V_n$ with the familiar Euclidean space $\mathbb{R}^n$. However, one shall keep in mind, $\mathbb{R}^n$, as a vector space equipped with an inner product, is more than a collection of arrays of real numbers. One should not think of a vector in $\mathbb{R}^n$ as an array of real numbers unless we specify a basis or a frame.

Tensor Product. For vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, the tensor product $b \otimes a$ is a linear mapping:

$$b \otimes a : V_n \to V_m$$

$$(b \otimes a)(c) = (a \cdot c)b \quad \forall c \in \mathbb{R}^n.$$ 

♦ Claim: For any $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, the mapping $b \otimes a$ (from $V_n$ to $V_m$) defined above is linear.

♦ Claim: Let $\{e_i : i = 1, \cdots n\}$ be an orthonormal basis of $\mathbb{R}^n$ and $\{\hat{e}_p : p = 1, \cdots m\}$ be an orthonormal basis of $\mathbb{R}^m$. Show that

$$\{\hat{e}_p \otimes e_i : i = 1, \cdots, n, p = 1, \cdots m\} \subset \text{Lin}(\mathbb{R}^n, \mathbb{R}^m)$$

forms a basis of the linear space Lin($\mathbb{R}^n, \mathbb{R}^m$).

Subspace of $\mathbb{R}^n$, Orthogonal Subspace. A subset $M \subset \mathbb{R}^n$ is a subspace if $\forall \alpha, \beta \in \mathbb{R} \& \forall a, b \in M$, $\alpha a + \beta b \in M$.

Let $M^\perp = \{b : b \cdot a = 0 \forall a \in M\}$. 

♦ Claim: Show that $M^\perp$ is a subspace of $\mathbb{R}^n$ if $M$ is a subspace.

Projection Theorem. Let $M$ be a subspace of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, we have

$$x = y + z$$

where $y \in M$, $z \in M^\perp$.

The vector $y$, $z$ are uniquely determined by $x$.

♦ Proof:
**Transpose of a Tensor** Let $A \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^m)$, $\{e_i : i = 1, \cdots, n\}$ be an orthonormal basis of $\mathbb{R}^n$ and $\{\hat{e}_p : p = 1, \cdots, m\}$ be an orthonormal basis of $\mathbb{R}^m$. Then $A$ admits the following decomposition

$$A = \sum_{p,i} A_{pi} \hat{e}_p \otimes e_i$$

where $A_{pi} = \hat{e}_p \cdot A(e_i) \quad \forall \; i = 1, \cdots, n, p = 1, \cdots, m$.

Define

$$A^T : \mathbb{R}_m^m \rightarrow \mathbb{R}^n,$$

$$A^T = \sum_{p,i} A_{pi} e_i \otimes \hat{e}_p \in \text{Lin}(\mathbb{R}^m, \mathbb{R}^n).$$

♦ Claim: For any $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$,

$$b \cdot A(a) = a \cdot A^T(b).$$

**Symmetric and Skew-symmetric Tensor** Let $A \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^n)$. $A$ is symmetric if $A = A^T$; $A$ is skew-symmetric if $A^T = -A$.

Let $\{e_i : i = 1, \cdots, n\}$, $\{\hat{e}_p : p = 1, \cdots, n\}$ be two orthonormal bases of $\mathbb{R}^n$. We have shown

$$A = \sum_{p,i} A_{pi} \hat{e}_p \otimes e_i$$

where $A_{pi} = \hat{e}_p \cdot A(e_i) \quad \forall \; p,i = 1, \cdots, n$.

♦ Claims:

1. For any $A \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^n)$, we have a unique decomposition $A = E + W$, where $E = E^T$ and $W = -W^T$.

2. $A = A^T$ if and only if for any $a, b \in \mathbb{R}^n$,

$$b \cdot A(a) = a \cdot A(b).$$

3. If $A = A^T$ and $a \cdot A(a) = 0$ for any $a \in \mathbb{R}^n$, then $A = 0$.

4. There exists a nonzero tensor $A$ such that

$$a \cdot Aa = 0 \quad \forall \; a \in \mathbb{R}^n, \; n \geq 2.$$  

5. Assume that $(\hat{e}_1, \cdots, \hat{e}_n) = (e_1, \cdots, e_n)$. If $A = A^T$, then $A_{pi} = A_{ip}$ for all $p, i = 1, \cdots, n$; if $A = -A^T$, then $A_{pi} = -A_{ip}$.

**Product of tensors** Let $A \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^m)$, $B \in \text{Lin}(\mathbb{R}^m, \mathbb{R}^k)$. Then

$$BA : \mathbb{R}^n \rightarrow \mathbb{R}^k,$$

$$BA(a) = B(A(a)).$$

**Orthogonal Tensor** Let $Q \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^n)$. The tensor $Q$ is orthogonal if $Qa \cdot Qb = a \cdot b$ for all $a, b \in \mathbb{R}^n$. From the definition we see that orthogonal tensor operating on vectors preserves the length of a vector and the angle between two vectors since
1. \( |a| = |Qa| \), and
2. \( a \cdot b = Qa \cdot Qb \).

\textbf{CLAIM:} A tensor \( Q : \mathbb{R}^n \to \mathbb{R}^n \) is orthogonal if and only if \( Q^TQ = QQ^T = I \), where \( I \) is the identity mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \).

**Trace and determinant of a tensor** Let \( A \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^n) \) and \( \{e_i : i = 1, \ldots, n\} \) be an orthonormal basis. Then we have \( A = \sum_{p,i} A_{pi} e_p \otimes e_i \) and refer to \( \text{Tr}(A) = \sum_{p=1}^n A_{pp} \) as the trace of \( A \), \( \det A = \det[A_{pi}] \) as the determinant of \( A \).

\textbf{CLAIM} \( \text{Tr}, \det : \text{Lin}(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R} \) is independent of the choice of basis.

**Rigid Rotation Tensor** An orthogonal tensor \( R \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^n) \) is a rigid rotation if \( \det R = +1 \).

**Representation theorem:** For any \( A \in \text{Lin}(\mathbb{R}^n, \mathbb{R}^n) \), there is an \( a \in \mathbb{R}^n \) such that Explicitly, if we have \( A = \sum_i A_{1i} \hat{e}_1 \otimes e_i \), \( \hat{e}_1 = 1 \),

then

\[ a = \sum_{i=1}^n A_{1i} e_i. \]

**Cross product in \( \mathbb{R}^3 \)** For \( a, b \in \mathbb{R}^3 \),

\[ a \wedge b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = W(b), \]

where \( W = \sum_{p,i} W_{pi} e_p \otimes e_i \),

\[ [W_{pi}] = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}. \]

\textbf{CLAIM:} The following properties of cross products holds:

1. \( b \wedge a = -a \wedge b \), \( a \cdot (a \wedge b) = 0 \), \( b \cdot (a \wedge b) = 0 \).
2. \( (a \wedge b) \cdot c = (b \wedge c) \cdot a = (c \wedge a) \cdot b \).
3. Geometric interpretation: show that \( |a \wedge b| = \text{area of the parallelogram formed by } a \text{ and } b \); \( |c \cdot (a \wedge b)| = \text{volume of the parallelepiped formed by } a, b, c \).