

Difference Equations and Symmetric Polynomials Defined by Their Zeros

FRIEDRICH KNOP & SIDDHARTHA SAHI*

Department of Mathematics, Rutgers University, New Brunswick NJ 08903, USA

knop@math.rutgers.edu, sahi@math.rutgers.edu

1. Introduction

In this paper, we are starting a systematic analysis of a class of symmetric polynomials which, in full generality, has been introduced in [Sa]. The main features of these functions are that they are defined by vanishing conditions and that they are non-homogeneous. They depend on several parameters but we are studying mainly a certain subfamily which is indexed by one parameter r . As a special case, we obtain for $r = 1$ the factorial Schur functions discovered by Biedenharn and Louck [BL].

Our main result is that for general r these functions are eigenvalues of difference operators, which are difference analogues of the Sekiguchi-Debiard differential operators. Thus the functions under investigation are non-homogeneous variants of Jack polynomials.

More precisely, let Λ be the set of partitions of length n , i.e., sequences of integers (λ_i) with $\lambda_1 \geq \dots \geq \lambda_n \geq 0$. The degree $|\lambda|$ of a partition λ is the sum of its parts. Choose a vector $\varrho \in \mathbb{C}^n$ which has to satisfy a mild condition. Then for every $\lambda \in \Lambda$ there is (up to a constant) a unique symmetric polynomial P_λ of degree at most d which satisfies the following vanishing condition:

$$P_\lambda(\mu + \varrho) = 0 \text{ for all partitions } \mu \text{ with } |\mu| \leq |\lambda| \text{ and } \mu \neq \lambda.$$

This kind of vanishing comes up in the study of invariant differential operators and Capelli type identities on multiplicity free spaces and has been, in special cases, observed by other authors (e.g. [HU], [Ok]).

In full generality, we have basically only one result (beyond their existence) about the polynomials P_λ , namely two explicit formulas for P_λ when $\lambda = 1^k$. From then on, we are only considering $\varrho = r\delta$, where $r \in \mathbb{C}$ and $\delta = (n - 1, n - 2, \dots, 1, 0)$.

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We prove that these P_λ are simultaneous eigenfunctions of n commuting *difference* operators. On the highest homogeneous part of a polynomial, these difference operators act like well known differential operators: the Sekiguchi-Debiard operators. The eigenfunctions of those are the Jack polynomials. This has as immediate consequence that the top homogeneous part of P_λ is a Jack polynomial.

In the later sections, we draw several conclusions from the difference equations. As an application to the “classical” theory we give a new proof of the Pieri rule for Jack polynomials using the polynomials P_λ .

We conclude with a brief discussion of the “integral” form J_λ which in the homogeneous case, is a rescaling of the P_λ by a certain hooklength factor. It turns out that the corresponding *inhomogeneous* polynomial seems to have integrality and positivity properties which generalize a conjecture of Macdonald for the homogeneous case. In this connection, we have recently proved some integrality and positivity results which we shall report on elsewhere.

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2. The basic construction

The results of this section are essentially in [Sa], however in order to keep the development self-contained we give a quick rederivation.

Let us write $S(n, d) \subset \mathbb{Z}^n$ for the set of partitions $\lambda_1 \geq \dots \lambda_n \geq 0$ with $|\lambda| := \sum \lambda_i = d$. We say that $\varrho \in \mathbb{C}^n$ is *dominant* if $\varrho_i - \varrho_j \neq -1, -2, -3, \dots$ for all $i < j$. Slightly weakening this condition, we define ϱ to be *d-dominant* if $\varrho_i - \varrho_j \neq -1, -2, -3, \dots, -\lfloor \frac{d}{i} \rfloor$ for all $i < j$ where $d \in \mathbb{N}$.

2.1. Theorem. *For any $d \in \mathbb{N}$ and $\varrho \in \mathbb{C}^n$ put $M := S(n, d) + \varrho \subseteq \mathbb{C}^n$. Assume, ϱ is d-dominant. Then for every map $\bar{f} : M \rightarrow \mathbb{C}$ there is a unique symmetric polynomial f of degree at most d such that $f|_M = \bar{f}$.*

Proof: For any partition $\lambda \in \mathbb{Z}^n$ let m_λ be the corresponding monomial symmetric function in n variables. If we express an arbitrary symmetric function of degree $\leq d$ in terms of m_λ , then the interpolation problem gives a *square* system of linear equations for the coefficients. Hence existence implies uniqueness.

To show existence, we argue by induction on $n + d$. The case $n = 0$ is vacuous, so we assume $n \geq 1$.

To any $\lambda \in S(n-1, d)$ we can append a zero and obtain a partition $\lambda, 0 \in S(n, d)$. This way, we can define map $g = \sum a_\lambda m_\lambda \mapsto g^+ = \sum a_\lambda m_{\lambda, 0}$. It is an injective map from symmetric functions in $n-1$ variables to symmetric functions in n variables. It has the property that g^+ has the same degree as g , and $g^+(x_1, \dots, x_{n-1}, 0) = g(x_1, \dots, x_{n-1})$.

We will construct f as a function of the form

$$f(x) = g^+(x_1 - \varrho_n, \dots, x_n - \varrho_n) + \left[\prod_{i=1}^n (x_i - \varrho_n) \right] h(x_1 - 1, \dots, x_n - 1)$$

First, let us consider the set M_0 of all points $x = \lambda + \varrho \in M$ with $\lambda_n = 0$. Since $x_n - \varrho_n = 0$, the first term equals $g(x_1 - \varrho_n, \dots, x_{n-1} - \varrho_n)$ and the second term vanishes. If x runs through M_0 then $x' = (x_1 - \varrho_n, \dots, x_{n-1} - \varrho_n)$ runs through $S(n-1, d) + \varrho'$, where $\varrho' := (\varrho_1 - \varrho_n, \dots, \varrho_{n-1} - \varrho_n)$ which is also d -dominant. By induction we can find g of degree $\leq d$ with $f(x) = g(x') = \bar{f}(x)$ for all $x \in M_0$.

Next, we consider the points $x \in M \setminus M_0$, i.e., $x = \lambda + \varrho \in M$ with $\lambda_n > 0$. These exist only if $d \geq n$. As x runs through these points $(x_1 - 1, \dots, x_n - 1)$ will run through $S(n, d-n) + \varrho$. Since $\lfloor d/i \rfloor \geq \lambda_i \geq \lambda_n > 0$ and since ϱ is d -dominant, each of the factors $x_i - \varrho_n = \lambda_i + \varrho_i - \varrho_n$ is non-zero. By induction, we can find h of degree $\leq d-n$ such that h has prescribed values at $M \setminus M_0$. \square

We assume from now on that ϱ is dominant. With the theorem, we are going to define interpolation polynomials. To get the most convenient normalization, we have to introduce some more notation: Recall that a partition λ can be represented by its *diagram*, i.e., the set of all lattice points (called boxes) $(i, j) \in \mathbb{Z}^2$ with $1 \leq i \leq n$ and $1 \leq j \leq \lambda_i$. The dual partition λ' is the one with the transposed diagram. Now, for every box s we define the ϱ -hooklength to be $c_\lambda^\varrho(s) := (\lambda_i - j + 1) + (\varrho_i - \varrho_{\lambda'_j})$ and $c_\lambda^\varrho := \prod_{s \in \lambda} c_\lambda^\varrho(s)$.

Definition: For any partition $\lambda \in S(n, d)$ let P_λ^ϱ be the unique polynomial in n variables such that

- (1) P_λ^ϱ is symmetric;
- (2) $\deg P_\lambda^\varrho = d$;
- (3) $P_\lambda^\varrho(\mu + \varrho) = 0$ for all $\mu \in S(n, d)$, $\mu \neq \lambda$;
- (4) $P_\lambda^\varrho(\lambda + \varrho) = c_\lambda^\varrho$.

The normalization condition (4) is motivated by the following theorem. In fact, we could replace (4) by it.

2.2. Theorem. Let $P_\lambda^\varrho = \sum_{\mu: |\mu| \leq |\lambda|} u_{\lambda\mu}^\varrho m_\mu$ be the expression in terms of monomial symmetric functions. Then $u_{\lambda\lambda}^\varrho = 1$.

Proof: We proceed by induction on $n + |\lambda|$. As in the proof of Theorem 2.1 we express

$$P_\lambda^\varrho = g^+(x_1 - \varrho_n, \dots, x_n - \varrho_n) + \left[\prod_{i=1}^n (x_i - \varrho_n) \right] h(x_1 - 1, \dots, x_n - 1)$$

First assume $\lambda_n = 0$. Put $\nu := (\lambda_1, \dots, \lambda_{n-1})$ and $\varrho' := (\varrho_1 - \varrho_n, \dots, \varrho_{n-1} - \varrho_n)$. Then Theorem 2.1 implies $g = aP_\nu^{\varrho'}$ with $a \in \mathbb{C}^*$. Now, we compare values at $x = \lambda + \varrho$. Since $c_\lambda^\varrho = c_\nu^{\varrho'}$ we obtain $a = 1$ and the assertion follows by induction.

Next, suppose $\lambda_n > 0$. Then Theorem 2.1 implies $g = 0$ and $h = aP_\nu^\varrho(x_1 - 1, \dots, x_n - 1)$ where $\nu := (\lambda_1 - 1, \dots, \lambda_n - 1)$ and $a \in \mathbb{C}^*$. Again, we compare values at $x = \lambda + \varrho$. The linear factors are just the ϱ -hooklengths for the first column of λ . Thus, $a = 1$ and the assertion follows by induction. \square

Additionally, we got the following reduction formula:

2.3. Corollary. *Assume λ is a partition with $\lambda_n > 0$ and let $\lambda^* := (\lambda_1 - 1, \dots, \lambda_n - 1)$. Then $P_\lambda^\varrho = \prod_i (x_i - \varrho_n) P_{\lambda^*}^\varrho(x_1 - 1, \dots, x_n - 1)$.*

3. Special cases

We don't know an explicit formula for P_λ^ϱ in general but several special cases are known.

For arbitrary ϱ we have only a formula for $\lambda = 1^k$. This is the partition with k ones and $(n - k)$ zeros. The functions $P_{1^k}^\varrho$ are important since they are analogues of the elementary symmetric functions. In particular, they generate the symmetric polynomials as a ring. Actually, we have *two* formulas for them.

Recall that the elementary symmetric function $e_j(x)$ and the complete symmetric function $h_j(y)$ are the coefficients of t^j in the expansions of $E(x, t) = \prod_i (1 + tx_i)$ and $H(y, t) = \prod_i (1 - ty_i)^{-1}$ respectively.

3.1. Proposition. *Let ϱ be dominant and $1 \leq k \leq n$. Then*

$$P_{1^k}^\varrho = \sum_{j=0}^k (-1)^{k-j} h_{k-j}(\varrho_k, \dots, \varrho_n) e_j(x) = \sum_{i_1 < \dots < i_k} \prod_{j=1}^k (x_{i_j} - \varrho_{i_j + k - j}).$$

Proof: Denote the first expression by P' , the second by P'' . We are going to show that they both satisfy the definition of $P_{1^k}^\varrho$. Both have certainly the right degree and m_{1^k} has the right coefficient.

For the vanishing condition (3), let $x = \mu + \varrho$ with $|\mu| \leq k$ and $\mu \neq 1^k$. This forces $\mu_k = \dots = \mu_n = 0$ and $x_k = \varrho_k, \dots, x_n = \varrho_n$. Observe that P' is precisely the coefficient of t^k in the power series expansion of $\prod_{i=1}^n (1 + tx_i) / \prod_{i=k}^n (1 + t\varrho_i)$. Evaluated at x , this

quotient becomes a *polynomial* of degree $< k$, and its k -th coefficient $P'(x)$ vanishes. As for P'' , the index i_k in its definition is at least k . Hence the factors for $j = k$ vanish at x which shows $P''(x) = 0$.

Finally, we have to show symmetry. This is trivial for P' but not quite for P'' . First let $n = 2$. Then

$$P''_{11} = (x_1 - \varrho_1) + (x_2 - \varrho_2); \quad P''_{12} = (x_1 - \varrho_2)(x_2 - \varrho_2)$$

which are certainly symmetric. Now let $n \geq 3$. To make the dependence on ϱ and k visible, we write $P'' = P''_k(x; \varrho)$. Furthermore, let x', ϱ' (resp. x'', ϱ'') equal x, ϱ where we dropped the last (resp. first) component. If we break the defining sum for P'' up according $i_k < n$ or $i_k = n$ we get

$$P''_k(x; \varrho) = P''_k(x'; \varrho') + (x_n - \varrho_n)P''_{k-1}(x'; \varrho'').$$

By induction we see that P'' is symmetric in x_1, \dots, x_{n-1} . If we break the sum up according $i_1 = 1$ or not we obtain

$$P''_k(x; \varrho) = P''_k(x''; \varrho'') + (x_1 - \varrho_k)P''_{k-1}(x''; \varrho'').$$

This shows that P'' is symmetric in x_2, \dots, x_n as well. □

Remarks: For $\varrho = r(n-1, \dots, 1, 0)$, the expression P' is essentially due to Wallach while that for P'' can be traced back to Capelli. The equality $P' = P''$ can be also proved directly by using the polynomials $e_k(x/y)$ of [M3] p.58.

For the rest of the paper we specialize to ϱ of the form $r\delta$ where r is a complex number or just an indeterminate and $\delta := (n-1, \dots, 1, 0)$. The dominance of ϱ means that $r \neq -p/q$ where p, q are integers such that $p, q \geq 1$, and $q < n$. We shall assume this from now on.

First we treat the case $r = 0$. For this we introduce the *falling factorial polynomials* $x^{\underline{m}} := x(x-1) \dots (x-m+1)$. The factorial monomial symmetric functions m_λ are obtained by replacing each monomial $x_1^{l_1} x_2^{l_2} \dots x_n^{l_n}$ in m_λ by the corresponding factorial monomial $x_1^{\underline{l_1}} x_2^{\underline{l_2}} \dots x_n^{\underline{l_n}}$. The following is obvious.

3.2. Proposition. *For $r = 0$, we have $P_\lambda^0 = m_\lambda$.*

For $r = 1$ we get the factorial Schur functions. (See [BL], [M2], and [Ol].) To define them we write $a_\delta(x)$ for the Vandermonde determinant $\det(x_i^{\delta_j}) = \prod_{i < j} (x_i - x_j)$. Then the next result seems to be due to Okounkov [Ok].

3.3. Proposition. For $r = 1$, we have

$$P_\lambda^\delta(x) = \frac{1}{a_\delta(x)} \det \left(x_i^{\lambda_j + \delta_j} \right).$$

Proof: Since $\det \left(x_i^{\lambda_j + \delta_j} \right)$ is a skew symmetric polynomial, its quotient by a_δ is a symmetric polynomial which is easily seen to have degree $|\lambda|$. Now let $\mu \neq \lambda$ and $|\mu| \leq |\lambda|$. Since $a_\delta(\mu + \delta) \neq 0$ for any partition μ , it remains only to prove the vanishing of $\det \left[(\mu_i + \delta_i)^{\lambda_j + \delta_j} \right] = \sum_\sigma (-1)^\sigma \prod_i (\mu_{\sigma(i)} + \delta_{\sigma(i)})^{\lambda_i + \delta_i}$.

If a, b are nonnegative integers then $a^b = 0$ unless $a \geq b$. So the σ -summand vanishes unless $\mu_{\sigma(i)} + \delta_{\sigma(i)} \geq \lambda_i + \delta_i$ for all i . Summing over i , we observe that $|\mu| \leq |\lambda|$ forces equality for each i , which implies $\sigma(\mu + \delta) = \lambda + \delta$. But this is not possible for $\mu \neq \lambda$. \square

Finally we consider the analogue of the complete symmetric functions, i.e., $P_d^{r\delta}$ where d stands for $(d, 0, \dots, 0)$.

3.4. Proposition. For $d \geq 0$ we have

$$P_d^{r\delta} = \binom{-r}{d}^{-1} \sum_{i_j} \prod_{j=1}^n \left[\binom{-r}{i_{j-1} - i_j} (x_j - r\delta_j - i_j)^{i_{j-1} - i_j} \right]$$

where the sum runs through all integer sequences $d = i_0 \geq i_1 \geq \dots \geq i_{n-1} \geq i_n = 0$.

Proof: Let p_d denote the right hand side. Obviously, it has the right degree d and the coefficient of x_1^d is one. Next we show that the vanishing condition holds. For this let $x = \mu + r\delta$ with $|\mu| \leq |\lambda|$ and $\mu \neq \lambda$. Then every summand of p_d is a multiple of $y_1(y_2 - 1) \dots (y_d - d + 1)$ where $y_1 = \dots = y_{i_{n-1}} = x_n - r\delta_n = \mu_n$, $y_{i_{n-1}+1} = \dots = y_{i_{n-2}} = \mu_{n-1}$ etc. In particular, the y_i are integers with $0 \leq y_1 \leq \dots \leq y_d \leq \mu_1$. Now assume that the product does not vanish, i.e., $y_i \neq i - 1$ for all i . Then we claim $y_i \geq i$ for all i . Indeed, $y_i \geq y_{i-1} \geq i - 1$ and $y_i \neq i - 1$ imply $y_i \geq i$. In particular, $\mu_1 \geq y_d \geq d$. But this is not possible for our choice of μ . This shows $p_d(x) = 0$.

Finally, we have to prove symmetry. We are considering the case $n = 2$ first. For this we need two basic facts about falling factorials: (1) $x^a(x-a)^b = x^{a+b}$ which is obvious and the Vandermonde identity (2) $(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}$. Letting $i_0 = d \geq i_1 = i \geq i_2 = 0$ we obtain that p_d is a multiple of

$$\sum_i \binom{-r}{d-i} (x_1 - r - i)^{d-i} \binom{-r}{i} x_2^i.$$

Applying identity (2) this becomes

$$\sum_{i,j} \frac{(d-i)!(-r)^{d-i}(-r)^i(-r-i)^{d-i-j}}{j!(d-i-j)!(d-i)!i!} x_1^j x_2^i.$$

Using (1), the coefficient becomes $(-r)^{\underline{d-i}}(-r)^{\underline{d-j}}/j!(d-i-j)!i!$, which implies symmetry for $p_d(x_1, x_2)$.

Now suppose that $n \geq 3$. Summing over $i = i_{n-1}$ first, we obtain

$$p_d(x) = \binom{-r}{d}^{-1} \sum_{i=0}^d \binom{-r}{d-i} \binom{-r}{i} x_n^i p_{d-i}(x_1 - r - i, \dots, x_{n-1} - r - i)$$

By induction we conclude that p_d is symmetric in $\{x_1, \dots, x_{n-1}\}$. Summing over $i = i_1$ we obtain

$$p_d(x) = \binom{-r}{d}^{-1} \sum_{i=0}^d \binom{-r}{d-i} \binom{-r}{i} (x_1 - r\delta_1 - i)^{\underline{d-i}} p_i(x_2, \dots, x_n)$$

which proves symmetry in $\{x_2, \dots, x_n\}$. This concludes the proof. \square

4. Difference operators and Jack polynomials

In this section we deduce a different characterization of the polynomials $P_\lambda^{r\delta}$ in terms of difference equations.

Let ε_i be the i -th canonical basis vector in \mathbb{C}^n . The i -th *shift operator* T_i on functions is defined by $T_i f(x) := f(x - \varepsilon_i)$, and the i -th *difference operator* is $\nabla_i := 1 - T_i$. These operators commute with each other, and T_i, ∇_i also commute with multiplication by x_j for $j \neq i$.

Definition: Let t be an indeterminate. For $1 \leq i, j \leq n$ put

$$\Delta_{ij} := (x_i + t)(x_i + r)^{\delta_j} - x_i^{\delta_j+1} T_i, \quad \Delta := \det(\Delta_{ij}), \quad \mathcal{D}(t; r) := a_\delta(x)^{-1} \Delta.$$

Since Δ_{ij} and Δ_{kl} commute for $i \neq k$, the determinant Δ is well defined. Furthermore, it maps symmetric polynomials to skew-symmetric ones. Hence $\mathcal{D}(t; r)$ is a well defined operator acting on the space of symmetric polynomials. We can develop

$$\mathcal{D}(t; r) = D_0 t^n + D_1 t^{n-1} + \dots + D_n$$

into a polynomial where D_i is a difference operator of order i and $D_0 = 1$.

4.1. Example. For $r = 0$ we obtain $\mathcal{D}(t; r) = (t + x_1 \nabla_1) \dots (t + x_n \nabla_n)$, hence $D_i = e_i(x_1 \nabla_1, \dots, x_n \nabla_n)$.

We need the following partial order relation on \mathbb{Z}^n : we say $\mu \leq \lambda$ if $\mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i$ for all $1 \leq i \leq n$. It has the property that λ is a partition if and only if it is maximal among all its permutations.

4.2. Lemma. *The operator $\mathcal{D}(t; r)$ is triangular. More precisely,*

$$\mathcal{D}(t; r)m_\lambda \in \prod_i (\lambda_i + r\delta_i + t)m_\lambda + \sum_{\mu < \lambda} \mathbb{C}[t]m_\mu.$$

In particular, $\deg \mathcal{D}(t; r)f \leq \deg f$ for every symmetric polynomial f .

Proof: The transition matrix between Schur function s_λ and monomial symmetric functions m_μ is unitriangular. Hence, it suffices to prove $\mathcal{D}(t; r)m_\lambda \in \prod_i (\lambda_i + r\delta_i + t)s_\lambda + \sum_{\mu < \lambda} \mathbb{C}[t]s_\mu$. Now we multiply by a_δ . By definition, $a_{\lambda+\delta} = a_\delta s_\lambda$ is the skew symmetrization of $x^{\lambda+\delta}$. Therefore, it suffices to prove that Δm_λ is a linear combination of monomials x^μ with $\mu \leq \lambda + \delta$ and that the coefficient of $x^{\lambda+\delta}$ has the indicated form.

For this, observe $\Delta_{ij} = x_i^{\delta_j} (x_i \nabla_i + r\delta_j + t) + \text{lower terms in } x_i$, and that $x_i \nabla_i (x_i^m) = mx_i^m + \text{lower terms}$. Thus

$$\Delta_{ij} x_i^m = (m + r\delta_j + t)x_i^{m+\delta_j} + \text{lower terms in } x_i.$$

Expanding the determinant defining Δ , we see that all monomials occurring in Δm_λ are of the form x^μ with $\mu = \sigma(\lambda) + \tau(\delta) - \eta$ where σ, τ are permutations and $\eta \in \mathbb{N}^n$. All these μ are $\leq \lambda + \delta$. Furthermore, $\mu = \lambda + \delta$ implies $\sigma(\lambda) = \lambda$, $\tau = 1$, and $\eta = 0$. In particular, only the diagonal term contributes to $x^{\lambda+\delta}$. Hence, we obtain

$$\Delta m_\lambda \in \prod_i (\lambda_i + r\delta_i + t)x^{\lambda+\delta} + \sum_{\mu < \lambda+\delta} \mathbb{C}[t]x^\mu. \quad \square$$

For $I \subseteq \{1, \dots, n\}$, put $\varepsilon_I := \sum_{i \in I} \varepsilon_i$, and $T_I f := (\prod_{i \in I} T_i) f = f(x - \varepsilon_I)$. Furthermore, we introduce the functions $\varphi_I(x) := \det c_{ij}^I(x)$ where

$$c_{ij}^I := \begin{cases} x_i^{\delta_j+1} & \text{for } i \in I; \\ (x_i + r)^{\delta_j} & \text{for } i \notin I. \end{cases}$$

They behave like ‘‘cut-off functions’’:

4.3. Lemma. *Let $r \neq 0$ and μ be a partition. If $\mu - \varepsilon_I$ is not a partition then $\varphi_I(\mu + r\delta) = 0$.*

Proof: Put $x = \mu + r\delta$ and assume $\mu - \varepsilon_I$ is not a partition. Then there are two cases:

- (1) $\mu_n = 0$ and $n \in I$. Then $x_n = 0$ and the n -th row of $c^I(x)$ vanishes. Hence $\varphi_I(x) = 0$.
- (2) There is $i < n$ such that $i \in I$, $i + 1 \notin I$, and $\mu_i = \mu_{i+1}$. In this case $x_i = x_{i+1} + r$ and c^I has two proportional rows. Hence, again $\varphi_I(x) = 0$ and the claim is proved. \square

Now we prove that each $P_\lambda^{r\delta}$ is an eigenfunction of $\mathcal{D}(t; r)$. More precisely:

4.4. Theorem. *For each partition λ , we have*

$$\mathcal{D}(t; r)P_\lambda^{r\delta} = \prod_i (\lambda_i + r\delta_i + t)P_\lambda^{r\delta}.$$

In particular, the action of $\mathcal{D}(t; r)$ on symmetric polynomials is diagonalizable with distinct eigenvalues.

Proof: In view of Lemma 4.2, it suffices to show that $\mathcal{D}(t; r)P_\lambda^{r\delta}$ satisfies the vanishing condition. We may exclude the case $r = 0$ either by direct computation or by continuity. Since then $a_\delta(\mu + r\delta) \neq 0$ for all partitions μ , we are left with $\Delta(f)$.

We can expand Δ as follows: $\Delta = \sum_I d_I T_I$, where $d_I = \det d_{ij}^I$ and

$$d_{ij}^I := \begin{cases} -x_i^{\delta_j+1} & \text{for } i \in I; \\ (x_i + t)(x_i + r)^{\delta_j} & \text{for } i \notin I. \end{cases}$$

Since d_I is a multiple of φ_I , Lemma 4.3 holds also for it. Let μ be a partition with $|\mu| \leq |\lambda|$, $\mu \neq \lambda$. Then $\Delta P_\lambda^{r\delta}(\mu + r\delta) = \sum_I d_I(\mu + r\delta)P_\lambda^{r\delta}(\mu - \varepsilon_I + r\delta)$. Since $P_\lambda^{r\delta}$ satisfies the vanishing condition it follows from Lemma 4.3 that $d_I(\mu + r\delta)P_\lambda^{r\delta}(\mu - \varepsilon_I + r\delta) = 0$ for all I . This finishes the proof of the vanishing condition for $\mathcal{D}(t; r)P_\lambda^{r\delta}$ and of the Theorem. \square

Since the $P_\lambda^{r\delta}$ form also an eigenbasis for D_1, \dots, D_n we obtain:

4.5. Corollary. *The difference operators D_1, \dots, D_n commute pairwise.*

4.6. Corollary. *Every $P_\lambda^{r\delta}$ has an expansion of the form $m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu$.*

Proof: Lemma 4.2 implies that $\mathcal{D}(t; r)$ preserves the finite dimensional space spanned by $\{m_\mu \mid \mu \leq \lambda\}$. Thus, by the theorem, it has an eigenvector with the above expansion, which by the lemma has the same eigenvalue as $P_\lambda^{r\delta}$. So, they are equal. \square

Now we can make the connection to the Jack polynomials. First, we recall their definition: for an indeterminate t consider the differential operators

$$\bar{\Delta} := \det \left(x_i^{\delta_j} \left(t + r\delta_j + x_i \frac{\partial}{\partial x_i} \right) \right); \quad \bar{\mathcal{D}}(t; r) := a_\delta^{-1} \bar{\Delta}.$$

These operators were introduced by Sekiguchi [Se] and Debiard [De]. Macdonald, [M1], uses them to define the Jack polynomial $P_\lambda^{(1/r)}$: it is the unique eigenvector of $\bar{\mathcal{D}}(t; r)$ which is of the form $m_\lambda + \sum_{\mu < \lambda} a_\mu m_\mu$.

4.7. Corollary. *The top homogeneous component of $P_\lambda^{r\delta}$ is $P_\lambda^{(1/r)}$.*

Proof: Denote this component by \bar{P} . As observed in the proof of Lemma 4.2 $\Delta_{ij} = x_i^{\delta_j} (x_i \nabla_i + r\delta_j + X) + \text{lower terms}$, and $x_i \nabla_i = x_i \frac{\partial}{\partial x_i} + \text{lower terms}$. Thus $\mathcal{D}(t; r)$ acts on \bar{P} by $a_\delta^{-1} \det(x_i^{\delta_j} (x_i \frac{\partial}{\partial x_i} + r\delta_j + t)) = \bar{\mathcal{D}}(t; r)$. Consequently \bar{P} is an eigenfunction of the Sekiguchi-Debiard operator. The assertion follows from Corollary 4.6. \square

5. The extra vanishing theorem

Corollary 4.6 states that $P_\lambda^{r\delta}$ contains less monomials than it could according to its definition. In this section we establish a property of $P_\lambda^{r\delta}$ which is in a way “dual” to that: we are going to prove that $P_\lambda^{r\delta}$ vanishes at more points than it should by definition.

Recall that $\lambda \subset \mu$ means $\lambda_i \leq \mu_i$ for all i , i.e., the diagrams are contained in each other. Let \mathcal{P} be the set of partitions. A subset S of \mathcal{P} is called *closed* if $\lambda \in S$, $\mu \in \mathcal{P}$ and $\lambda \subset \mu$ implies $\mu \in S$. For every closed set S we consider the ideal \mathcal{I}_S of symmetric polynomials which vanish at all point $\mu + r\delta$ where μ a partition which is *not* in S .

5.1. Theorem. *Let $S \subseteq \mathcal{P}$ be closed. Then the ideal \mathcal{I}_S is stable under the action of $\mathcal{D}(t; r)$.*

Proof: Again, we may exclude $r = 0$ by continuity. Then we have to show that $\Delta(f)(x) = 0$ whenever $f \in \mathcal{I}_S$ and $x = \mu + r\delta$ with $\mu \in \mathcal{P} \setminus S$. As in the proof of Theorem 4.4 it suffices to consider the products $\varphi_I(x)f(x - \varepsilon_I)$. Assume this does not vanish. Then $\mu' = \mu - \varepsilon_I \in \mathcal{P}$ with $f(\mu' + r\delta) \neq 0$. But then $\mu' \in S$, and therefore $\mu \in S$ contradicting the choice of μ . \square

Now we can prove the extra vanishing theorem:

5.2. Theorem. *Let λ and μ be partitions with $\lambda \not\subset \mu$. Then $P_\lambda^{r\delta}(\mu + \varrho) = 0$.*

Proof: Consider the closed subset S of all μ containing λ . We have to show $P_\lambda^{r\delta} \in \mathcal{I}_S$. Now for generic r , there exist functions in \mathcal{I}_S which are *non-zero* at $\lambda + r\delta$. (For example, the product of falling factorials $\prod_{i,j,k} (x_i - r\delta_j)^{\lambda_k}$ is such a function). The ideal \mathcal{I}_S is $\mathcal{D}(t; r)$ -stable. Since $\mathcal{D}(t; r)$ is diagonalizable, there must be an eigenfunction of $\mathcal{D}(t; r)$ in \mathcal{I}_S with this property. But this function must be a multiple of some $P_\mu^{r\delta}$. Then $P_\mu^{r\delta}(\lambda + r\delta) \neq 0$ implies $|\mu| \leq |\lambda|$. Since $P_\mu^{r\delta}(\mu + r\delta) \neq 0$ we have $\lambda \subset \mu$. Hence $\mu = \lambda$. \square

This can be extended:

5.3. Corollary. *Let $S \subseteq \mathcal{P}$ be closed. Then $\mathcal{I}_S = \bigoplus_{\lambda \in S} \mathbb{C}P_\lambda^{r\delta}$.*

Proof: Since \mathcal{I}_S is \mathcal{D} -stable, there must be $S' \subseteq \mathcal{P}$ with $\mathcal{I}_S = \bigoplus_{\lambda \in S'} \mathbb{C}P_\lambda^{r\delta}$. Let $\lambda \in S'$. Since $P_\lambda^{r\delta}(\lambda + r\delta) \neq 0$, it can not be in $\mathcal{P} \setminus S$. Hence $S' \subseteq S$. Conversely, let $\lambda \in S$ and assume there is $\mu \in \mathcal{P} \setminus S$ with $P_\lambda^{r\delta}(\mu + r\delta) \neq 0$. Then $\lambda \subset \mu$ by the extra vanishing theorem. Hence $\mu \in S$ which is impossible. This shows $S \subseteq S'$. \square

To round this discussion off, let us mention the following

5.4. Proposition. *Let Λ be the ring of symmetric polynomials (in n variables). Then every \mathcal{D} -stable ideal of Λ is of the form \mathcal{I}_S for some closed subset S of \mathcal{P} .*

Proof: Clearly, every \mathcal{D} -stable ideal is of the form $\bigoplus_{\lambda \in S} \mathbb{C}P_\lambda^{r\delta}$. We have to show that S is closed. For this we need the following weak form of Pieri's rule proved in the next section: let $e_1 = \sum_i x_i$. Expand $e_1 P_\lambda^{r\delta} = \sum_\mu a_\mu P_\mu^{r\delta}$. Then $a_\mu \neq 0$ whenever $\mu = \lambda + \varepsilon_i \in \mathcal{P}$. This implies $\mu = \lambda + \varepsilon_i \in S$ whenever $\lambda \in S$ and $\mu \in \mathcal{P}$ which is equivalent to S being closed. \square

6. The dehomogenization operators and the Pieri formula

Both the $P_\lambda^{r\delta}$ and the Jack polynomials $P_\lambda^{(1/r)}$ form a basis of the algebra Λ of symmetric polynomials. In particular, there is a linear isomorphism $\Psi : \Lambda \rightarrow \Lambda$ which maps $P_\lambda^{(1/r)}$ to $P_\lambda^{r\delta}$. We are going to show that Ψ can also be described in terms of difference operators.

For this we define the following variant of \mathcal{D} :

$$\mathcal{E} := a_\delta^{-1} \det[(x_i + r)^{\delta_j} + tx_i^{\delta_j+1} T_i] = 1 + \mathcal{E}_1 t + \dots + \mathcal{E}_n t^n.$$

Let $\Lambda_d \subseteq \Lambda$ be the subspace spanned by all $P_\lambda^{r\delta}$ with $|\lambda| = d$. This is also the space of all polynomials of degree $\leq d$ which vanish in all $\mu + r\delta$ with $|\mu| \leq d - 1$.

6.1. Lemma. *We have $\mathcal{E}_k(\Lambda_d) \subseteq \Lambda_{d+k}$. Moreover, the effect of \mathcal{E}_k on the top homogeneous components is multiplication by the elementary symmetric function e_k .*

Proof: In the notation of section 4, \mathcal{E}_k has the expansion $\mathcal{E}_k = a_\delta^{-1} \sum_{|I|=k} \varphi_I T_I$. Hence $\mathcal{E}_k f(x) = a_\delta^{-1}(x) \sum_{|I|=k} \varphi_I(x) f(x - \varepsilon_I)$. Let $f \in \Lambda_d$ and μ be a partition with $|\mu| \leq d+k-1$ and $x = \mu + r\delta$. Then we have $\varphi_I(x) f(x - \varepsilon_I) = 0$. This means $\mathcal{E}_k f \in \Lambda_{d+k}$.

For the top homogeneous terms, $T_I = 1$ and $\varphi_I = \prod_{i \in I} x_i$, hence \mathcal{E}_k acts like multiplication by e_k . \square

Now we can prove

6.2. Theorem. *a) The difference operators $\mathcal{E}_1, \dots, \mathcal{E}_n$ commute pairwise.*

b) Let $\psi : \Lambda \rightarrow \mathbb{C}[\mathcal{E}_1, \dots, \mathcal{E}_n]$ be the isomorphism with $\psi(e_k) = \mathcal{E}_k$. Then $\Psi(f) = \psi(f)(1)$ (evaluation at 1) for all $f \in \Lambda$.

Proof: Let $\Lambda_{(d)}$ be the space of symmetric homogeneous polynomials of degree d . Then $\Psi : \Lambda_{(d)} \xrightarrow{\sim} \Lambda_d$ and the inverse is given by taking the top homogeneous component. Thus Lemma 6.1 implies that the following diagram commutes

$$\begin{array}{ccc} \Lambda_{(d)} & \xrightarrow{\Psi} & \Lambda_d \\ \downarrow e_k & & \downarrow \mathcal{E}_k \\ \Lambda_{(d+k)} & \xrightarrow{\Psi} & \Lambda_{d+k} \end{array}$$

Hence $\Psi(e_k f) = \mathcal{E}_k \Psi(f)$ for all $f \in \Lambda$. This shows a). Let $f(x) = p(e_1, \dots, e_k)$. Then $\Psi(f) = \Psi(p(e_k)) = p(\mathcal{E}_k) \Psi(1) = \psi(f)(1)$. \square

As an application of the theory above we give a new proof of the Pieri rule for Jack polynomials.

At each lattice point $s = (i, j)$ in the diagram of λ , the *lower* and *upper* hook-lengths are defined by $c_\lambda(s) = c_\lambda(\alpha; s) := \alpha(\lambda_i - j) + (\lambda'_j - i + 1)$, and $c'_\lambda(s) = c'_\lambda(\alpha; s) := \alpha(\lambda_i - j + 1) + (\lambda'_j - i)$.

Let $\mu \subset \lambda$. Then $X(\lambda/\mu)$ denotes the set of all boxes $(i, j) \in \lambda$ such that $\mu_i = \lambda_i$ and $\mu'_j < \lambda'_j$. Then we define

$$\psi'_{\lambda/\mu}(\alpha) := \prod_{s \in X(\lambda/\mu)} \frac{c_\lambda(\alpha; s)/c'_\lambda(\alpha; s)}{c_\mu(\alpha; s)/c'_\mu(\alpha; s)}.$$

The Pieri formula is the following identity:

6.3. Theorem. *For every partition λ holds $e_k P_\mu^{(\alpha)} = \sum_\lambda \psi'_{\lambda/\mu}(\alpha) P_\lambda^{(\alpha)}$ where λ runs over all partitions of the form $\mu + \varepsilon_I$ for some $I \subset \{1, \dots, n\}$ with $|I| = k$, i.e. $\lambda - \mu$ is a vertical k -strip.*

Proof: Applying Ψ to both sides, it suffices to prove $\mathcal{E}_k P_\mu^{r\delta} = \sum_\lambda \psi'_{\lambda/\mu}(1/r) P_\lambda^{r\delta}$, summed over $\{\lambda \mid \lambda - \mu \text{ is a vertical } k\text{-strip}\}$. In any case, $\mathcal{E}_k P_\mu^{r\delta} = \sum_\lambda a_{\lambda\mu} P_\lambda^{r\delta}$ where λ is a partition of degree $|\mu| + k$. Evaluating at the point $x = \lambda + r\delta$ and using the expansion of \mathcal{E}_k we see $a_{\lambda\mu} P_\lambda^{r\delta}(\lambda + r\delta) = \mathcal{E}_k P_\mu^{r\delta}(x) = a_\delta(\lambda + r\delta)^{-1} \varphi_I(\lambda + r\delta) P_\mu^{r\delta}(\mu + r\delta)$. Hence, it remains to prove the identity

$$\psi'_{\lambda/\mu}(1/r) = a_\delta(\lambda + r\delta)^{-1} \varphi_I(\lambda + r\delta) (c_\lambda^{r\delta})^{-1} c_\mu^{r\delta}.$$

We first calculate $c_\lambda^{r\delta}/c_\mu^{r\delta} = r^{|\lambda| - |\mu|} c'_\lambda/c'_\mu$. Let us put $I' := \{i \notin I\}$, $J := \{\lambda_i \mid i \in I\}$ and $J' = \{\lambda_i \mid i \in I'\}$, and for simplicity, let us write $c'_\lambda(i, j)$ instead of $c'_\lambda(1/r; (i, j))$. Then it is easy to see that for $i \in I$, we have $c'_\lambda(i, j + 1) = c'_\mu(i, j)$ unless $j \in J'$. Similarly, for $i \in I'$ $c'_\lambda(i, j) = c'_\mu(i, j)$ unless $j \in J$. Taking these cancelations into account we get

$$\frac{c_\lambda^{r\delta}}{c_\mu^{r\delta}} = \frac{r^{|\lambda|} c'_\lambda}{r^{|\mu|} c'_\mu} = r^k \prod_{i \in I} c'_\lambda(i, 1) \prod_{i \in I, j \in J'} \frac{c'_\lambda(i, j + 1)}{c'_\mu(i, j)} \prod_{i \in I', j \in J} \frac{c'_\lambda(i, j)}{c'_\mu(i, j)}.$$

On the other hand, $a_{\delta}^{-1}(\lambda + r\delta)\varphi_I(\lambda + r\delta)$ equals

$$\prod_{i \in I} (\lambda_i + r\delta_i) \prod_{\substack{i \in I, k \in I' \\ i < k}} \frac{(\lambda_i + r\delta_i) - (\lambda_k + r\delta_k + r)}{(\lambda_i + r\delta_i) - (\lambda_k + r\delta_k)} \prod_{\substack{i \in I, k \in I' \\ k < i}} \frac{(\lambda_k + r\delta_k + r) - (\lambda_i + r\delta_i)}{(\lambda_k + r\delta_k) - (\lambda_i + r\delta_i)}$$

Now the set $\{k \in I' \mid \lambda_k = 0\}$ equals $\{\lambda'_1 + 1, \lambda'_1 + 2, \dots, n\}$, and for $j \in J'$, we have $\{k \in I' \mid \lambda_k = j\} = \{\lambda'_{j+1} + 1, \lambda'_{j+1} + 2, \dots, \mu'_j\}$. Thus the first two products, which can be rewritten as $\prod_{i \in I} (\lambda_i + r(n - i)) \prod_{i \in I, k \in I', i < k} \frac{\lambda_i - \lambda_k + r(k - i - 1)}{\lambda_i - \lambda_k + r(k - i)}$, become after cancelation,

$$\prod_{i \in I} (\lambda_i + r(\lambda'_1 - i)) \prod_{\substack{i \in I, j \in J' \\ (i, j) \in \mu}} \frac{\lambda_i - j + r(\lambda'_{j+1} - i)}{\lambda_i - j + r(\mu'_j - i)} = r^k \prod_{i \in I} c'_\lambda(i, 1) \prod_{i \in I, j \in J'} \frac{c'_\lambda(i, j + 1)}{c'_\mu(i, j)}.$$

Finally, for each $j \in J$, the set $\{i \in I \mid \lambda_i = j\}$ equals $\{\mu'_j + 1, \mu'_j + 2, \dots, \lambda'_j\}$. Thus after cancelation the third product $\prod_{j \in J, k \in I', k < i} \frac{\lambda_k - \lambda_i + r(i - k + 1)}{\lambda_k - \lambda_i + r(i - k)}$ becomes

$$\prod_{\substack{j \in J, k \in I' \\ (k, j) \in \mu}} \frac{\lambda_k - j + r(\lambda'_j - k + 1)}{\lambda_k - j + r(\mu'_j - k + 1)} = \prod_{i \in I', j \in J} \frac{c_\lambda(i, j)}{c_\mu(i, j)}.$$

Since $\psi'_{\lambda/\mu}(1/r) = \prod_{i \in I', j \in J} \frac{c_\lambda(i, j)/c'_\lambda(i, j)}{c_\mu(i, j)/c'_\mu(i, j)}$, the result follows. \square

7. Scholium.

We close with a conjecture on the “integral” form of the Jack polynomial. In the homogeneous case, this is the function $J_\lambda^{(\alpha)} = c_\lambda(\alpha)P_\lambda^{(\alpha)}$. In the inhomogeneous situation, consider the function:

$$J_\lambda^{r\delta}(x) := (-1)^{|\lambda|} c_\lambda(1/r) P_\lambda^{r\delta}(-x).$$

Various computations suggest the following extension of a conjecture of Macdonald for J_λ^α .

Conjecture. Put $\alpha = 1/r$, and write $J_\lambda^{r\delta} = \sum_{\mu \leq \lambda} \alpha^{|\mu| - |\lambda|} a_{\lambda\mu}(\alpha) m_\mu$. Then $a_{\lambda\mu}$ is a polynomial in α with positive integral coefficients.

Recently we have proved Macdonald’s original conjecture as well as the integrality part of the above conjecture. We shall report on these developments elsewhere.

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