

The Asymptotic Behavior of Invariant Collective Motion

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For Armand Borel

1. Introduction

Let G be a connected complex reductive group acting on a smooth algebraic variety X . Then the cotangent bundle T_X^* on X carries a canonical symplectic structure, and the G -action induces a moment map $\Phi : T_X^* \rightarrow \mathfrak{g}^*$. Consider the Hamiltonian vector fields attached to functions of the form $f \circ \Phi$ with $f \in \mathbb{C}[\mathfrak{g}^*]^G$. In this paper we study the asymptotic behavior of the associated flow (a so-called invariant collective motion) and show that it possesses a symmetry with respect to a finite reflection group W_X . This is applied to the theory of equivariant embeddings of X . The approach is purely algebraic.

More specifically: Choose any generic point $\alpha \in T_X^*$. Because the functions $f \circ \Phi$ with $f \in \mathbb{C}[\mathfrak{g}^*]^G$ are in involution (i.e., their Poisson product vanishes), the flow through α is in the orbit of an abelian group A_α . It is known (see [GS]) that this orbit is also the orbit for the connected isotropy group $G_{\Phi(\alpha)}^0$. This implies that A_α is a linear algebraic group and it turns out that it is a torus. The projection of this orbit to X is called a *flat* of X and just equals $G_{\Phi(\alpha)}^0 \pi(\alpha)$. In case, X is the complexification of a symmetric space, a flat in our sense is the complexification of a usual flat (=maximal totally geodesic, flat submanifold).

Let $X \subseteq \bar{X}$ be a normal equivariant embedding. The main point of this paper is to study the closure of a generic flat in \bar{X} . This will be done in two different steps.

The first one is to show that a certain finite group W_X acts on them. Consider the family of tori $\alpha \mapsto A_\alpha$. Although every two of these groups are isomorphic to each other, the family cannot in general be trivialized globally. But we show that it can be trivialized on a finite cover \widehat{T}_X of an open subset of T_X^* . Hence, there is an action of a torus A_X

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on \widehat{T}_X such that flats are just the projections of the orbits to X . The map of \widehat{T}_X onto its image in T_X^* is a Galois covering such that its group W_X is a subquotient of the Weyl group W of G . This group W_X also acts on A_X (Theorem 4.1, Theorem 4.2).

Let $\overline{F}_{\widehat{\alpha}}$ be the closure of $\pi(A_X \widehat{\alpha})$ in \overline{X} where $\widehat{\alpha} \in \widehat{T}_X$. Observe that $\overline{F}_{\widehat{\alpha}} = \overline{F}_{w\widehat{\alpha}}$ for every $w \in W_X$. There is a rigidity lemma (Lemma 6.1) which says that for any torus action the closures of any two generic orbits are canonically isomorphic. Applied to our situation, this gives an isomorphism $\overline{F}_{\widehat{\alpha}} \xrightarrow{\sim} \overline{F}_{w\widehat{\alpha}} = \overline{F}_{\widehat{\alpha}}$, which implies:

1.1. Theorem. *For $\widehat{\alpha} \in \widehat{T}_X$ generic, the action of W_X on $A_X \cong F_{\widehat{\alpha}}$ can be extended to $\overline{F}_{\widehat{\alpha}}$. [Corollary 6.3]*

The second part is as follows: For each A_X -orbit $E \subseteq \overline{F}_{\widehat{\alpha}}$ consider the set of one-parameter subgroups $\lambda : \mathbf{G}_m \rightarrow A_X$ such that $\lim_{t \rightarrow 0} \pi(\lambda(t)\widehat{\alpha})$ exists and is contained in E . Then let $\mathcal{C}(E)$ be the closure of the convex cone generated by this set inside $\mathcal{H}(X) := \text{Hom}(\mathbf{G}_m, A_X) \otimes \mathbb{Q}$. Because $\overline{F}_{\widehat{\alpha}}$ contains only finitely many orbits, we get a finite set of cones, a so-called fan. It is known that the normalization of $\overline{F}_{\widehat{\alpha}}$ is completely determined by its fan.

The main point is that we are able to control the limit behavior of one-parameter subgroups which are in a certain cone $\mathcal{Z}(X) \subseteq \mathcal{H}(X)$. To describe this cone, let $\mathcal{V}(X)^G$ be the set of all G -invariant discrete valuations (with values in \mathbb{Q}) of the field of rational functions $K := \mathbb{C}(X)$. For a Borel subgroup $B \subseteq G$ let $K^{(B)}$ be the multiplicative group of B -semiinvariant rational functions, i.e., for any $f \in K^{(B)}$ there is a character $\chi_f \in \mathcal{X}(B)$ with $f^b = \chi_f(b)f$ for all $b \in B$. One can prove that each $v \in \mathcal{V}(X)^G$ is completely determined by its restriction to $K^{(B)}$. Hence, $\mathcal{V}(X)^G$ may be identified with a subset of $\text{Hom}(K^{(B)}, \mathbb{Q})$.

Now we define $\mathcal{Z}(X)$ to be the set of those invariant valuations, which restrict to the trivial valuation on K^B . These valuations are called *central*. Then $\mathcal{Z}(X)$ can be viewed as a subset of $\mathcal{H}(X)$ as follows: Let $\Gamma_X := \{\chi_f \mid f \in K^{(B)}\} = K^{(B)}/(K^*)^B$. This is a subgroup of $\mathcal{X}(B)$, hence finitely generated free abelian. It follows from the definition, that $\mathcal{Z}(X)$ can be regarded as a subset of $\text{Hom}(\Gamma_X, \mathbb{Q})$. Next, one shows that Γ_X is canonically isomorphic to the character group $\mathcal{X}(A_X)$, so one gets

$$\mathcal{Z}(X) \hookrightarrow \text{Hom}(\Gamma_X, \mathbb{Q}) = \text{Hom}(\mathcal{X}(A_X), \mathbb{Q}) \cong \text{Hom}(\mathbf{G}_m, A_X) \otimes \mathbb{Q} = \mathcal{H}(X).$$

Having this, we prove:

1.2. Theorem. *Let $\lambda : \mathbf{G}_m \rightarrow A_X$ be the one-parameter subgroup of A_X . Assume it corresponds to a valuation $v \in \mathcal{Z}(X)$. Let Y be the center of v in \overline{X} . Then for $\widehat{\alpha} \in \widehat{T}_X$ generic, the limit $\lim_{t \rightarrow 0} \pi(\lambda(t)\widehat{\alpha})$ exists and is contained in Y . [Theorem 7.3]*

The main idea of proof is the use of a slice theorem, essentially due to Brion-Luna-Vust [BLV], which describes the local structure of \bar{X} around Y as a B -variety.

These two theorems above combine to give a full picture of the limit behavior of one-parameter subgroups because we can prove:

1.3. Theorem. *The cone $\mathcal{Z}(X)$ is a fundamental domain for the group W_X , i.e., every orbit intersects $\mathcal{Z}(X)$ in exactly one point.* [Theorem 7.4]

This is the main theorem of this paper. Its proof is roughly as follows: It is known that $\mathcal{Z}(X) \subseteq \mathcal{H}(X)$ is a finitely generated convex cone. Let W'_X be the group generated by the reflections at the codimension-one-faces of $\mathcal{Z}(X)$. Then general functorial properties of W_X and W'_X imply $W'_X \subseteq W_X$, and in particular, that $\mathcal{Z}(X)$ meets every W_X -orbit at least once. On the other hand, the first two theorems imply that every W_X -orbit intersects $\mathcal{Z}(X)$ in at most one point. As a corollary we get that $W_X = W'_X$ is generated by reflections and that $\mathcal{Z}(X)$ is a simplicial cone.

If X is spherical, i.e., X contains an open B -orbit, then $K^B = \mathbb{C}$ which implies that every invariant valuation is central. That means that W_X completely describes $\mathcal{V}(X)^G$. But also in the general case, W_X determines at least the qualitative structure of $\mathcal{V}(X)^G$: It is the union of simplicial cones which intersect along their common face $\mathcal{Z}(X)$ (see [Kn4] §9 for details).

In this paper only the third theorem is proved in full generality, while the first two are only proved for varieties which I have named *non-degenerate*. There are three reasons to justify this: “Most” (e.g. all quasiaffine) varieties are non-degenerate. There are techniques (affine cones, Kostant’s shifted cotangent bundles) to reduce the general case to the non-degenerate one. Finally, the proof in the general case is so much more involved that it would have almost doubled the size of this paper. But I want to mention that I have a proof for the general case which I will publish elsewhere.

The surprising fact, that $\mathcal{Z}(X)$ is the fundamental domain of a finite group was, for spherical X , first proved by Brion [Br], but his construction of W_X was not very enlightening. The present paper is the result of my effort for a more geometric construction of W_X . It has been announced in [Kn1], a paper which contains the basic results on the moment map, while [Kn4] does the same for invariant valuations. Unfortunately, the preparation of [Kn4] took quite a time, during which I was able to simplify some proofs in [Kn1]. Therefore, the present paper is quite independent of [Kn1] since it reproves most results in the non-degenerate case. It replaces the never published preprint [Kn2].

Notation: All varieties are defined over an algebraically closed field k of characteristic zero. The group G is always reductive and connected. We choose a Borel subgroup $B \subseteq G$ with unipotent radical U and maximal torus T . The Lie algebra of any group is usually

denoted by the corresponding fraktur letter. The character group of a group H (or Lie algebra \mathfrak{h}) is denoted by $\mathcal{X}(H)$ (or $\mathcal{X}(\mathfrak{h}) = (\mathfrak{h}^*)^{\mathfrak{h}}$). If $H \rightarrow GL(V)$ is a representation then $V^{(H)}$ is the set of H -eigenvectors. The character for $v \in V^{(H)}$ is χ_v . If H acts on an affine variety X then $X//H := \text{Spec } k[X]^H$.

2. The local structure theorem

In this section we present a refined version of the local structure theorem of Brion-Luna-Vust [BLV]. Let X be a normal, but not necessarily smooth G -variety. A B -divisor is a formal linear combination $D = \sum a_i D_i$ where the D_i are B -stable prime Cartier divisors of X and the coefficients are elements of k . If they are in \mathbb{Z} or \mathbb{N} then we call D integral or effective, respectively. The support of D is the union of those D_i with $a_i \neq 0$. We also will denote it with D , when no confusion is possible. The stabilizer of the support of D is denoted by $P[D]$. It is a parabolic subgroup of G .

Being Cartier, each D_i defines a line bundle $\mathcal{O}(D_i)$ on X with a canonical section σ_i . There is a finite covering \tilde{G} of G such that every $\mathcal{O}(D_i)$ can be \tilde{G} -linearized. We will, for convenience of notation, replace G by \tilde{G} . The sections σ_i are eigenvectors for $P[D]$ and hence determine infinitesimal characters $\chi_{D_i} \in \mathcal{X}(\mathfrak{p}[D])$. We define $\chi_D := \sum_i a_i \chi_{D_i}$. Observe, that the linearizations are unique up to a character of G . Therefore, also χ_D is only well defined up to a character of \mathfrak{g} .

For any parabolic subgroup P let Δ_P^+ be the set of roots in the unipotent radical of P . Consider a character $\chi \in \mathcal{X}(\mathfrak{p})$. Then we call it P -regular, if $\langle \chi, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Delta_P^+$. A B -divisor D is regular, if χ_D is $P[D]$ -regular. There are plenty of them:

2.1. Lemma. *Every effective B -divisor is regular.*

Proof: Let D be an effective B -divisor. It induces a line bundle $\mathcal{L} = \mathcal{O}(D)$ with a section σ . We may assume that \mathcal{L} is G -linearized. Then, $P[D]$ is clearly just the normalizer of the line $k\sigma$ and χ_D is the character with which $\mathfrak{p}[D]$ acts on it. Because the action of G on $H^0(X, \mathcal{L})$ is locally finite, σ is a highest weight vector and χ_D is an integral dominant weight. Hence the roots of the unipotent radical of $P[D]$ are just those α such that $\langle \chi_D, \alpha^\vee \rangle > 0$. \square

If τ is another section of $\mathcal{O}(D_i)$ then $\frac{\tau}{\sigma_i}$ is a regular function on $X \setminus D$. Hence, we can define a morphism

$$\psi_D : X \setminus D \longrightarrow \mathfrak{g}^* : x \mapsto l_x \quad \text{where} \quad l_x(\xi) := \sum_i a_i \frac{\xi \sigma_i}{\sigma_i}(x),$$

which is well-defined up to a translation by a character of \mathfrak{g} .

If D is an effective B -divisor, then it induces directly a line bundle $\mathcal{O}(D)$ with section σ_D , character χ_D and morphism ψ_D . Note, that our previous definitions are compatible with these notions.

Later on, there may occur divisors which are not Cartier. The next lemma tells us that the morphism ψ_D is defined anyway because we may replace X by $X \setminus \bigcap_{g \in G} gD$.

2.2. Lemma. *Let X be a normal G -variety and $D \subseteq G$ a prime divisor. Then, D is a Cartier divisor outside $Z = \bigcap_{g \in G} gD$.*

Proof: We may assume that Z is empty. Let $\iota : X_r \hookrightarrow X$ be the inclusion of the subset of smooth points of X . Because $D \cap X_r$ is a Cartier divisor, it defines a line bundle \mathcal{L}_r on X_r . There is a finite cover $\tilde{G} \twoheadrightarrow G$ such that the action of \tilde{G} on X_r can be lifted to a linearization of \mathcal{L}_r . Therefore, \tilde{G} acts also on $\mathcal{L} := \iota_* \mathcal{L}_r$. Because X is normal, \mathcal{L} is a (trivial) line bundle on $X \setminus D$. Hence it is a line bundle everywhere, because Z is empty by assumption. But that means precisely that D is a Cartier divisor. \square

Now we are in the position to state the local structure theorem:

2.3. Theorem. *Let X be a normal G -variety with a B -divisor $D = \sum_i a_i D_i$. Assume that χ_D is $P[D]$ -regular. Then:*

- a) *The image of ψ_D is a single $P[D]$ -orbit with isotropy subgroup a Levi complement of $P[D]$.*
- b) *For some $x_0 \in X \setminus D$ let*

$$\eta_0 := \psi_D(x_0), \quad L := G_{\eta_0}, \quad \Sigma := \psi_D^{-1}(\eta_0).$$

Then L is a Levi subgroup of $P[D]$ and there is an isomorphism

$$P[D] \times^L \Sigma \longrightarrow X \setminus D$$

Proof: a) Let \mathfrak{l}_0 be a Levi complement of $\mathfrak{p} = \mathfrak{p}[D]$, with center \mathfrak{z} and semisimple part \mathfrak{l}'_0 . We will identify \mathfrak{g}^* with \mathfrak{g} by means of an invariant scalar product.

For $x \in X \setminus D$ let $\eta := \psi_D(x)$. Because all σ_i are $P[D]$ -eigenvectors, we have $\eta \in (\mathfrak{l}'_0 \oplus \mathfrak{p}_u)^\perp = \mathfrak{z} \oplus \mathfrak{p}_u$. Because the eigencharacter is χ_D , we actually have

$$\text{Im } \psi_D \subseteq \xi_D + \mathfrak{p}_u.$$

Here, ξ_D is the element in \mathfrak{z} which is dual to $\chi_D \in \mathfrak{z}^*$, i.e., $(\xi_D, \xi) = \chi_D(\xi)$ holds for all $\xi \in \mathfrak{l}_0$. Therefore, the $P[D]$ -regularity of χ_D implies, that \mathfrak{l}_0 is the centralizer of ξ_D in \mathfrak{g} . It further follows that $\xi_D + \mathfrak{p}_u$ is a single $P[D]$ -orbit.

b) By a) also $L = G_{\eta_0}$ is a Levi subgroup of $P[D]$ and ψ_D induces a $P[D]$ -morphism $X \setminus D \rightarrow P[D]/L$. But that is equivalent to $P[D] \times^L \Sigma \rightarrow X \setminus D$ being an isomorphism. \square

Because of $P[D \cup D'] = P[D] \cap P[D']$ there exists a B -divisor D such that $P[D]$ is absolutely minimal. Let $P(X) := P[D]$ and $P_u(X)$ its unipotent radical. Hence, $P(X)$ is the largest subgroup $H \subseteq G$ such that every B -stable divisor is H -stable. It is easy to see ([Kn4] 2.5) that the definition is compatible with that in [Kn1]:

$$P(X) = \{g \in G \mid gBx = Bx \text{ for } x \in X \text{ generic}\}.$$

2.4. Proposition. *Let D be regular B -divisor with $P[D] = P(X)$ and let L, Σ as in Theorem 2.3. Then the commutator subgroup L' acts trivially on Σ .*

Proof: Let $B_L := B \cap L$. Then B_L -stable divisors in Σ correspond to B -stable divisors in X . Hence, they are L -stable. By [Su] there is a dense open L -stable subset $\Sigma_0 \subseteq \Sigma$ which can be equivariantly embedded into a projective space $\mathbf{P}(V)$. Consider the closure of the affine cone over Σ_0 in V . Let Σ' be its normalization. It has the property that every $B_L \times k^*$ -stable divisor is $L \times k^*$ -stable. Because Σ' is normal, this implies that every highest weight vector in $k[\Sigma']$ is an eigenvector for L . Therefore, L' acts trivially on $k[\Sigma']$ and hence on Σ' (being affine), Σ_0 and Σ . \square

Consider a situation as in the local structure theorem. Let $L_0 \subseteq L$ be the kernel of the action on Σ , and $S(X) := L_0 P_u(X)$. Then, the orbits of $U, S(X)$ and $P_u(X)$ on $X \setminus D$ coincide, the latter acting freely. Furthermore, $A_X := L/L_0 = P(X)/S(X)$ is a torus acting effectively on Σ . Because $k(X)^{(B)} = k(\Sigma)^{(L)}$ we get for the character group

$$\mathcal{X}(A_X) = \{\chi_f \mid f \in k(X)^{(B)}\}.$$

This shows that A_X does not depend on the particular choice of D .

3. Polarized cotangent vectors

Assume that X is smooth. Then the cotangent bundle $\pi : T_X^* \rightarrow X$ is a vector bundle. The G -action induces the *moment map*

$$\Phi : T_X^* \longrightarrow \mathfrak{g}^* : \alpha \mapsto l_\alpha \quad \text{where} \quad l_\alpha(\xi) = \alpha(\xi_{\pi(\alpha)}).$$

Assume that D is principal B -divisor, i.e., $D = \sum_j b_j [f_j]$ where $b_j \in k$ and $[f_j]$ is the principal divisor attached to a rational function $f_j \in k(X)^{(B)}$. Then the map ψ_D factors through the moment map: In fact, define

$$\psi_D^* : X \setminus D \longrightarrow T_X^* : x \mapsto \sum_j b_j f_j(x)^{-1} (df_j)_x.$$

Then $\xi f(x) = (df)_x(\xi)$ implies $\psi_D = \Phi \circ \psi_D^*$. It is easily checked that ψ_D^* is $P[D]$ -equivariant. Note that it can be expressed formally as $\psi_D^* = d \log \prod_j f_j^{b_j}$.

This map will be our link between the geometries of X and T_X^* . Unfortunately, there are cases when there exists no regular principal divisor with $P[D] = P(X)$, e.g., for $X = G/B$ every principal B -divisor is trivial but $P(X) = B$. This leads to the following

Definition: A G -variety X is called *non-degenerate* if there is a principal B -divisor D such that χ_D is $P(X)$ -regular.

Note, that $P(X)$ -regularity of χ_D implies $P[D] = P(X)$. In fact, $P(X)$ is contained in $P[D]$ by minimality and contains it by regularity. There are many non-degenerate varieties, e.g.:

3.1. Lemma. *Every quasiaffine G -variety is non-degenerate.*

Proof: Let D be any B -divisor of a quasiaffine G -variety X with $P[D] = P(X)$. Then there is $f \in k[X]^{(B)}$ vanishing on the support of D . Hence, $P[[f]] = P(X)$ and $[f]$ is regular because it is effective. \square

Let $f_1, \dots, f_s \in k(X)^{(B)}$ be a transcendence basis of $k(X)^U$ and let $D_0 := \cup_j [f_j]$. For any s -tuple $\underline{b} \in k^s$ let $D(\underline{b}) := \sum_j b_j [f_j]$. Then we get the map

$$\psi^* : (X \setminus D_0) \times \mathbf{A}^s \longrightarrow T_X^* : (x, \underline{b}) \mapsto \psi_{D(\underline{b})}^*(x).$$

Suppose X is non-degenerate. Then all U -orbits in $X \setminus D_0$ will have the same dimension by Proposition 2.4. Consider

$$X_0 := \{x \in X \setminus D_0 \mid (df_1)_x, \dots, (df_s)_x \text{ are linearly independent}\}.$$

Because the f_i form a transcendence basis of $k(X)^U$ this set is not empty and for any $x \in X_0$ the common kernel of the $(df_j)_x$ is the tangent space to the orbit Ux . Hence, ψ^* induces an isomorphism of $X_0 \times \mathbf{A}^s$ onto

$$C := \Phi^{-1}(\mathfrak{u}^\perp) \cap \pi^{-1}(X_0) = \{\alpha \in T_X^* \mid x = \pi(\alpha) \in X_0 \text{ and } \alpha(\mathfrak{u}x) = 0\}.$$

In particular, we get

$$\text{codim } C = \dim T_X^* - (\dim X + \text{tr.deg } k(X)^U) = \dim X - \dim \Sigma = \dim P_u(X).$$

Before I proceed let me set up some notation. Sometimes we will identify \mathfrak{g} and \mathfrak{g}^* by means of an invariant scalar product. Let L be a Levi complement of $P = P(X)$ and let L_0 be the intersection of L with $S = S(X)$. We identify the Lie algebra of A_X with the orthogonal complement \mathfrak{a} of \mathfrak{l}_0 in \mathfrak{l} . Let $P^- = P(X)^-$ be the parabolic opposite to P

with $P \cap P^- = L$. Let $\mathfrak{a}^r \subseteq \mathfrak{a}^*$ be the set of P -regular elements. It is the complement of finitely many hyperplanes.

3.2. Theorem. *Let X be non-degenerate. Then $G \cdot C$ is dense in T_X^* .*

Proof: By the local structure theorem, the generic U - and S -orbits in X coincide. Hence, $\Phi(C) \subseteq \mathfrak{s}^\perp = \mathfrak{a} \oplus \mathfrak{p}_u$. Because \mathfrak{a}^* is spanned by the characters χ_{f_i} , the composition $C \rightarrow \mathfrak{a} \oplus \mathfrak{p}_u \rightarrow \mathfrak{a}$ is surjective. Via P_u , every element of $\mathfrak{a}^r \times \mathfrak{p}_u$ can be moved into \mathfrak{a}^r . This shows that the generic P_u -orbit of C contains an element α with $\xi := \Phi(\alpha) \in \mathfrak{a}^r$.

Consider the map between tangent spaces $d\Phi : T_\alpha(T_X^*) \rightarrow \mathfrak{g}$. Then $d\Phi(T_\alpha(C)) \subseteq \mathfrak{a} \oplus \mathfrak{p}_u$ while $d\Phi(\mathfrak{p}_u^- \alpha) = [\mathfrak{p}_u^-, \xi] = \mathfrak{p}_u^-$. The last equality again holds because ξ is regular. Hence, C and $P_u^- \alpha$ intersect transversally in α . This shows $\dim P_u^- \cdot C = \dim P_u^- + \dim C = \dim T_X^*$, which implies the assertion. \square

3.3. Corollary. *Let X be non-degenerate. Then $\overline{\Phi(T_X^*)} = \overline{G \cdot \mathfrak{a}^*}$.*

This recovers [Kn1] 5.4 for non-degenerate varieties. Note that, by the same theorem, the last two statements are definitely wrong for degenerate varieties.

The Weyl group W of \mathfrak{g} acts also on \mathfrak{t}^* . Let $N(\mathfrak{a}^*) := \{w \in W \mid w\mathfrak{a}^* = \mathfrak{a}^*\}$ and let $W(\mathfrak{a}^*)$ be the image of $N(\mathfrak{a}^*)$ in the automorphism group of \mathfrak{a}^* . Then the map $\mathfrak{a}^r \rightarrow \mathfrak{t}^*/W$ is unramified and its image can be identified with $\mathfrak{a}^r/W(\mathfrak{a}^*)$. Consider the following diagram:

$$\begin{array}{ccccc}
& & T_X^* \times_{\mathfrak{t}^*/W} \mathfrak{a}^r & \xrightarrow{\widehat{\Phi}} & G/L \times \mathfrak{a}^r & \longrightarrow & \mathfrak{a}^r \\
& \widehat{\psi} \nearrow & \downarrow & & 1 \downarrow & & \downarrow \\
X_0 \times \mathbf{A}_r^s & \xrightarrow{\psi^*} & T_X^* & \xrightarrow{\Phi} & \overline{G\mathfrak{a}^*} & \xrightarrow{2} & \mathfrak{t}^*/W
\end{array}$$

Here, the arrow 1 is the map $(gL, \xi) \mapsto g\xi$ and arrow 2 uses the Chevalley isomorphism $\mathfrak{g}^*/G \rightarrow \mathfrak{t}^*/W$. It is easy to see that the right and hence also the left hand square is cartesian. Finally, the main point is that ψ^* lifts to $\widehat{\psi}$ by setting $\widehat{\psi}(x, \underline{b}) = (\psi^*(x), \chi_{D(\underline{b})})$. Here $\mathbf{A}_r^s \subseteq \mathbf{A}^s$ is the preimage of \mathfrak{a}^r by the surjective linear map $\mathbf{A}^s \rightarrow \mathfrak{a}^* : \underline{b} \mapsto \chi_{D(\underline{b})} = \sum_j b_j \chi_{f_j}$.

The set $Z := T_X^* \times_{\mathfrak{t}^*/W} \mathfrak{a}^r$ is smooth, but in general not connected. Thus, $\widehat{\psi}$ singles out a specific component which we denote by \widehat{T}_X . Its elements are called *polarized cotangent vectors*. The group $W(\mathfrak{a}^*)$ permutes the components of Z transitively. The subgroup $W_X \subseteq W(\mathfrak{a}^*)$ of those elements which map \widehat{T}_X into itself is called the *little Weyl group* of X . It acts freely on \widehat{T}_X and the map $\widehat{T}_X/W_X \rightarrow T_X^*$ is an open embedding.

A little Weyl group has been already defined in [Kn1]. These definitions are compatible:

3.4. Lemma. *Let X be non-degenerate. Then W_X is the same as defined in [Kn1].*

Proof: To see this, choose a linear section τ of $\mathbf{A}^s \rightarrow \mathfrak{a}^*$ and $x \in X_0$ and let

$$\widehat{\sigma} : \mathfrak{a}^r \xrightarrow{x \times \tau} X_0 \times \mathbf{A}_r^s \xrightarrow{\widehat{\psi}} \widehat{T}_X.$$

This is a section of $\widehat{\Psi} : \widehat{T}_X \rightarrow \mathfrak{a}^r$ which composed with $\widehat{T}_X \rightarrow T_X^*$ gives the map σ in [Kn1] 6.2. The existence of $\widehat{\sigma}$ implies that the generic fibers of $\widehat{\Psi}$ and therefore also of $\widehat{T}_X/W_X \rightarrow \mathfrak{a}^r/W_X$ are connected. This implies that $k[\mathfrak{a}^r/W_X]$ is integrally closed in $k[T_X^*]$, hence equals the variety L_X of [Kn1] 6. This shows the claim. \square

4. Flats

Consider the composed map $\widehat{T}_X \rightarrow G/L \times \mathfrak{a}^r \rightarrow G/L$ and let $\widehat{\Sigma}$ and \widehat{C} be the preimage of eL and $P_u eL$ respectively. Then we have

$$\widehat{C} = P \times^L \widehat{\Sigma} \quad \text{and} \quad \widehat{T}_X = G \times^P \widehat{C} = G \times^L \widehat{\Sigma}.$$

The set $P_u eL \times \mathfrak{a}^r \subseteq G/L \times \mathfrak{a}^*$ is mapped isomorphically onto $\mathfrak{a}^r \times \mathfrak{p}_u \subseteq \mathfrak{g}^*$. Hence, the map $\widehat{C} \cap \widehat{\pi}^{-1}(X_0) \rightarrow C \subseteq T_X^*$ is an open embedding (where $\widehat{\pi} : \widehat{T}_X \rightarrow X$ is the canonical projection). This implies that

$$\widehat{\psi} : X_0 \times \mathbf{A}_r^s \hookrightarrow \widehat{C}$$

is an open embedding. Because $\widehat{\psi}^{-1}(\widehat{\Sigma}) \rightarrow \mathbf{A}_r^s$ is a parameterized family of slices as in Theorem 2.3, the Levi group L acts also on $\widehat{\Sigma}$ only via its quotient A_X . In particular, the generic isotropy group of \widehat{T}_X and that of T_X^* is conjugated to L_0 . This is a new proof of [Kn1] 8.2 for non-degenerate varieties.

Because A_X is abelian *this induces an A_X -action on $\widehat{T}_X = G \times^L \widehat{\Sigma}$ by $a \cdot [g, \widehat{\alpha}] := [g, a\widehat{\alpha}]$. It commutes with the action of G , but not with that of W_X as we will see later. There is another description of the A_X -action which does not refer to a particular choice of L : By means of $\widehat{\Phi}$ each element $\widehat{\alpha} \in \widehat{T}_X$ determines a coset $gL \in G/L$, hence a homomorphism*

$$\varphi_{\widehat{\alpha}} : G_{\widehat{\Phi}(\widehat{\alpha})} = gLg^{-1} \xrightarrow{\sim} L \rightarrow A_X.$$

Note, that $\varphi_{\widehat{\alpha}}$ does not depend on the particular choice of g in its coset. Then we have the formula

$$a \cdot \widehat{\alpha} = \bar{a}\widehat{\alpha} \quad \text{with any lift } \bar{a} \text{ of } a \text{ in } G_{\widehat{\Phi}(\widehat{\alpha})}.$$

Of course, it suffices to check this for $\widehat{\alpha} \in \widehat{\Sigma}$ where it is just the definition. It also shows $\ker \varphi_{\widehat{\alpha}} \subseteq G_{\widehat{\alpha}}$ with equality for generic $\widehat{\alpha}$. Thus we get a homomorphism

$$\Theta_{\widehat{\alpha}} : A_X \rightarrow G_{\widehat{\Phi}(\widehat{\alpha})}/G_{\widehat{\alpha}}$$

which is an isomorphism for generic $\widehat{\alpha}$.

Let A be any connected group acting on a symplectic variety Z and let $\Phi : Z \rightarrow \mathfrak{a}^*$ be a morphism. Then every $\xi \in \mathfrak{a}$ induces two vector fields on Z : The first one ξ_* induced by the A -action. The second is the Hamiltonian vector field H_{l_ξ} attached to the function $l_\xi(x) = \Phi(x)(\xi)$. Then Φ is called a moment map if these two vector fields coincide: $\xi_* = H_{l_\xi}$. The existence of a moment map has (among others) two important consequences:

1. Every Hamiltonian vector field is an infinitesimal symplectomorphism. Hence, the A -action preserves the symplectic structure of Z .
2. The A -action is uniquely determined by Φ .

We apply this to \widehat{T}_X which carries as an étale cover of T_X^* a canonical symplectic structure.

4.1. Theorem. *The morphism $\widehat{\Psi} : \widehat{T}_X \rightarrow \mathfrak{a}^*$ is a moment map for the A_X -action on \widehat{T}_X . In particular, this action is symplectic.*

Proof: Without loss of generality let $\widehat{\alpha} \in \widehat{\Sigma}$. Choose $\xi \in \mathfrak{a}$ which, considered as an element of $\mathfrak{l} \subseteq \mathfrak{g}$, we denote by $\bar{\xi}$. Consider the commutative diagram

$$\begin{array}{ccccc} \widehat{T}_X & \xrightarrow{\widehat{\Phi}} & G/L \times \mathfrak{a}^r & \longrightarrow & \mathfrak{a}^r \\ \downarrow & & \downarrow & & \\ T_X^* & \xrightarrow{\Phi} & \mathfrak{g}^* & & \end{array}$$

By definition of the A_X -action we have $\xi_{\widehat{\alpha}} = \bar{\xi}_{\widehat{\alpha}}$. The well known fact that Φ is a moment map implies $\bar{\xi}_{\widehat{\alpha}} = (H_{l_{\bar{\xi}}})_{\widehat{\alpha}}$. Hence we have to show $(H_{l_{\bar{\xi}}})_{\widehat{\alpha}} = (H_{l_\xi})_{\widehat{\alpha}}$ or equivalently $(dl_{\bar{\xi}})_{\widehat{\alpha}} = (dl_\xi)_{\widehat{\alpha}}$. This follows from the fact that ξ and $\bar{\xi}$ (considered as linear functions on \mathfrak{a}^* and \mathfrak{g}^* , respectively) induce the same covector in the point $(eL, \widehat{\Psi}(\widehat{\alpha}))$ of $G/L \times \mathfrak{a}^r$. \square

4.2. Theorem. *There is a W_X -action on A_X which is compatible with that on $\mathfrak{a}_X = \text{Lie } A_X$. With this action we have*

$$w(a \cdot \widehat{\alpha}) = wa \cdot w\widehat{\alpha} \quad \text{for all } w \in W_X, a \in A_X, \widehat{\alpha} \in \widehat{T}_X,$$

i.e., there is an action of $(W_X \times A_X) \times G$ on \widehat{T}_X .

Proof: Let $w \in W_X$ and $\widehat{\alpha} \in \widehat{T}_X$ generic. Because the actions of W_X and G commute we have $G_{w\widehat{\alpha}} = G_{\widehat{\alpha}}$ and $G_{\widehat{\Phi}(w\widehat{\alpha})} = G_{\widehat{\Phi}(\widehat{\alpha})}$. Now we define the action of w on A_X by

$$A_X \xrightarrow{\Theta_{\widehat{\alpha}}} G_{\widehat{\Phi}(\widehat{\alpha})}/G_{\widehat{\alpha}} = G_{\widehat{\Phi}(w\widehat{\alpha})}/G_{w\widehat{\alpha}} \xrightarrow{\Theta_{w\widehat{\alpha}}^{-1}} A_X.$$

This automorphism is independent of $\widehat{\alpha}$ by the rigidity of automorphisms of a torus ([Sp] 2.5.10) and the irreducibility of \widehat{T}_X . Let $\bar{a} \in G_{\widehat{\Phi}(\widehat{\alpha})} \subseteq G$ be a lift of $\Theta_{\widehat{\alpha}}(a)$. Then, by definition, it is also a lift of $\Theta_{w\widehat{\alpha}}(wa)$. Hence we get

$$wa \cdot w\widehat{\alpha} = \bar{a}w\widehat{\alpha} = w\bar{a}\widehat{\alpha} = w(a \cdot \widehat{\alpha}).$$

That the two actions on \mathfrak{a}_X are the same follows from Theorem 4.1: $w(\xi_*) = wH_{l_\xi} = H_{l_{w\xi}}$. \square

Definition: Let $\widehat{\pi} : \widehat{T}_X \rightarrow X$ be the projection and $\widehat{\alpha} \in \widehat{T}_X$. Then $F_{\widehat{\alpha}} := \widehat{\pi}(A_X \cdot \widehat{\alpha})$ is called a *flat* of X and the map $\zeta_{\widehat{\alpha}} : A_X \rightarrow F_{\widehat{\alpha}} : a \mapsto \widehat{\pi}(a \cdot \widehat{\alpha})$ a *polarization* of it.

Observe that $F_{w\widehat{\alpha}} = F_{\widehat{\alpha}}$, i.e., as a set $F_{\widehat{\alpha}}$ depends only on the image $\alpha \in T_X^*$. In fact, it can also be described entirely in terms of T_X^* as $F_{\widehat{\alpha}} = \pi(G_{\Phi(\alpha)}\alpha)$.

5. Twisted flats

Next we will describe how to deal with degenerate varieties. For this and for another purpose (proof of Theorem 7.4) we need the following

5.1. Theorem. *Let $p : \widetilde{X} \rightarrow X$ be an G -equivariant principal H -bundle, where H is a torus. Assume that $k(\widetilde{X})^B = k(X)^B$, i.e., H acts trivially on $k(\widetilde{X})^B$. Then*

1. $P(X) = P(\widetilde{X})$.
2. *There is a canonical exact sequence*

$$1 \longrightarrow H \longrightarrow A_{\widetilde{X}} \longrightarrow A_X \longrightarrow 1$$

3. *Assume \widetilde{X} is non-degenerate. Then, the little Weyl group $W_{\widetilde{X}}$ fixes H pointwise. In particular it acts on A_X .*

Proof: Let $X_0 = P(X) \times^L \Sigma$ as in the Local Structure Theorem 2.3 and put $\widetilde{\Sigma} := p^{-1}(\Sigma)$ and $\widetilde{X}_0 := p^{-1}(X_0) = G \times^L \widetilde{\Sigma}$. Choose $\widetilde{x} \in \widetilde{\Sigma}$ generic and let $x = p(\widetilde{x})$. Then the orbit map $g \mapsto g\widetilde{x}$ induces a morphism $\omega : L_x \rightarrow p^{-1}(x) = H\widetilde{x} = H$. Because H is a torus and $\omega(1) = 1$, this map is a homomorphism on L_x^0 (see [KKV] 1.2). In particular, $L' \subseteq L_x^0$ acts trivially on $\widetilde{\Sigma}$. This implies $P(\widetilde{X}) = P(X)$.

Because X and \widetilde{X} have the same field of B -invariants, the image of ω is dense, hence equals H . This implies $H = H\widetilde{x} \subseteq L\widetilde{x} = A_{\widetilde{X}}$ with quotient $Lx = A_X$. This shows the exact sequence. It is easy to see that the H -action on $\widetilde{\Sigma}$ coincides with that induced by $H \subseteq A_{\widetilde{X}}$. This implies the commutativity of the diagram

$$\begin{array}{ccc} \widehat{T}_{\widetilde{X}} & \longrightarrow & \mathfrak{a}_{\widetilde{X}}^* \\ \downarrow & & \downarrow \\ T_{\widetilde{X}}^* & \longrightarrow & \mathfrak{h}^* \end{array}$$

Because all maps are $W_{\widetilde{X}}$ -equivariant, the little Weyl group acts trivially on $\mathfrak{h} = \text{Lie } H$. \square

Let now X be any smooth but possibly degenerate variety. Choose an effective B -divisor D with $P[D] = P(X)$. It is regular by Lemma 2.1. Replace it by a multiple such that $\mathcal{O}(D)$ carries a G -linearization. Then $p : L := \text{Spec}_X \oplus_{n \in \mathbb{Z}} \mathcal{O}(nD) \rightarrow X$ is

the geometric realization of the line bundle $\mathcal{O}(D)$ where the zero-section is removed. In particular, it is a principal k^* -bundle. Let σ_D be the canonical section of $\mathcal{O}(D)$. Then $\sigma_D \in k[L]$, hence $p^{-1}(D)$ is a principal B -divisor. This shows that L is non-degenerate.

A little bit more generally, we choose once and for all a principal H -bundle $p : \tilde{X} \rightarrow X$, where H is a torus and such that \tilde{X} is non-degenerate. To force $k(\tilde{X})^B = k(X)^B$, we replace G by the product $G \times H$, the second factor acting trivially on X . We define $W_X := W_{\tilde{X}}$. We will see later (Corollary 7.5) that this definition is independent of the choice of \tilde{X} and is compatible with that in [Kn1].

Any $\hat{\alpha} \in \hat{T}_{\tilde{X}}$ gives rise to a map $A_{\tilde{X}} \rightarrow X : a \mapsto p \circ \hat{\pi}(a \cdot \hat{\alpha})$. Because it is H -invariant, it factors through $A_X = A_{\tilde{X}}/H$.

Definition: This map $\zeta_{\hat{\alpha}} : A_X \rightarrow X$ is called a *twisted (polarized) flat* of X . The map $\zeta_* : \hat{T}_{\tilde{X}} \times A_X \rightarrow X$ is the *general flat* (with respect to \tilde{X}).

On a degenerate variety there are no untwisted flats. But even for non-degenerate varieties there may be more twisted flats than ordinary ones: Let $G = GL_2(k)$, $T_0 := \text{diag}(*, 1)$ and $T := \text{diag}(*, *)$. Put $\varphi : \tilde{X} = G/T_0 \rightarrow X = G/T$. Then a flat of X through the base point is the orbit of a generic isotropy group of $\mathfrak{t}^\perp = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ which is just $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Hence, there is only one flat. But $\mathfrak{t}_0^\perp = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$. So the isotropy subgroup may be every maximal torus of G , i.e., any orbit of a one parameter subgroup of G is a twisted flat.

6. The closure of a flat

We are interested in the closure of flats. For this we need the following rigidity lemma. Recall, that an action of a connected algebraic group on a variety X is called *locally linear* if every point of X has a stable open neighborhood which can be equivariantly embedded into a projective space. Normality of X implies local linearity by a result of Sumihiro [Su]. Therefore, any stable subvariety of a normal variety is locally linear.

6.1. Lemma. *Let A be a torus, Z a locally linear A -variety, $\varphi : Z \rightarrow S$ an A -invariant morphism and $\sigma : S \rightarrow Z$ a section of φ . Assume, that S is irreducible and that for every $s \in S$ the orbit $A\sigma(s)$ is open and dense in the fiber $Z_s := \varphi^{-1}(s)$. Then there is a unique non-empty open subset $S' \subseteq S$ which is maximal with the following property:*

Choose any $s_0 \in S'$. Then there is a unique A -equivariant S -isomorphism

$$Z \times_S S' \xrightarrow{\sim} Z_{s_0} \times S'$$

such that $\sigma(s)$ is mapped for all $s \in S'$ to $(\sigma(s_0), s)$.

Proof: Uniqueness is clear because $A\sigma(s)$ is dense in Z_s , for every $s \in S$. For that reason, the union of all S' with the property above has this property. Hence there exists a unique maximal such subset if there exists any.

It remains to show existence. Because A is a torus and the action is locally linear, Z can be covered by A -stable affine open subsets ([Su]). Hence, again by uniqueness, we may assume that Z is affine. Consider the quotient $Z//A = \text{Spec } k[Z]^A$. Then φ factors through $Z//A$ and $Z//A \rightarrow S$ is birational. Because of the section σ it is an isomorphism, i.e., $k[S] = k[Z]^A$.

For a character $\chi \in \mathcal{X}(A)$, let $k[Z]_\chi$ be the corresponding space of eigenfunctions. Let $\mathcal{M} := \{\chi \in \mathcal{X}(A) \mid k[Z]_\chi \neq 0\}$. This is a finitely generated monoid. For $\chi \in \mathcal{M}$ choose any non-zero $g \in k[Z]_\chi$. Let $g_0(z) := g(\sigma\varphi(z))$. Then $g_0 \in k[Z]^A$ and $f_\chi := g/g_0 \in k(Z)_\chi$ is a rational function with the property $f_\chi|_{\sigma(S)} \equiv 1$. Because $A\sigma(s)$ is dense in Z_s , the function f_χ is uniquely determined by this property, hence independent of the choice of g . For the same reason, we have $f_\chi f_\eta = f_{\chi+\eta}$ for all $\chi, \eta \in \mathcal{M}$. Because \mathcal{M} is finitely generated, we can replace S and Z by affine open subsets, such that all f_χ are regular functions.

Let $g \in k[Z]_\chi$. Because every fiber of φ contains a dense orbit, we have $k(Z)^A = k(S)$. Hence $g/f_\chi \in k(S)$. But then it is actually regular, because $(g/f_\chi)|_{\sigma(S)} = g|_{\sigma(S)}$ is. This shows that $k[S] \rightarrow k[Z]_\chi : g \mapsto gf_\chi$ is an isomorphism for every $\chi \in \mathcal{M}$. Hence we get an isomorphism $k[S] \otimes_k k[\mathcal{M}] \rightarrow k[Z]$ or equivalently a trivialization $Z \rightarrow S \times F$ where F is the torus embedding $\text{Spec } k[\mathcal{M}]$. \square

6.2. Corollary. *Let X be a smooth G -variety and $X \hookrightarrow \bar{X}$ a normal equivariant embedding. Then there exists a unique equivariant embedding $A_X \hookrightarrow \bar{A}_{\bar{X}}$ (the closure of a generic twisted flat) and a non-empty open subset $S' \subseteq \hat{T}_{\bar{X}}$ such that the following holds:*

There is a unique closed embedding $S' \times \bar{A}_{\bar{X}} \rightarrow S' \times X$ such that every $(\hat{\alpha}, a) \in S' \times A_X$ is mapped to $(\hat{\alpha}, \zeta_{\hat{\alpha}}(a))$.

Furthermore, S' can be chosen to be stable under G , $A_{\bar{X}}$ and W_X .

Proof: Let Z_0 be the closure of the image of

$$\hat{T}_{\bar{X}} \times A_X \longrightarrow \hat{T}_{\bar{X}} \times X : (\hat{\alpha}, a) \mapsto (\hat{\alpha}, \zeta_{\hat{\alpha}}(a))$$

Then there is a non-empty open subset $S \subseteq \hat{T}_{\bar{X}}$ such that the fibers of $Z_0 \rightarrow \hat{T}_{\bar{X}}$ are, over S , exactly the closures of flats. The section is given by $\sigma(\hat{\alpha}) = (\hat{\alpha}, p \circ \hat{\pi}(\hat{\alpha}))$. Then the existence of S' follows from the preceding lemma. It is easily checked that Z_0 is stable under G , $A_{\bar{X}}$, and W_X , and that φ is equivariant. Hence, if we choose S' to be maximal, then it is unique and stable under all three groups. \square

6.3. Corollary. *The action of W_X on A_X extends to $\bar{A}_{\bar{X}}$.*

Finally, let me explain the case of a symmetric variety $X = G/H$. Here G is semisimple and H is the set of fixed points of an involution $\vartheta \in \text{Aut } G$. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be the decomposition into eigenspaces of ϑ . Then \mathfrak{h} is the Lie algebra of H and \mathfrak{p} can be identified with \mathfrak{h}^\perp , hence $T_X^* = G \times^H \mathfrak{p}$. Choose a generic $\xi \in \mathfrak{p}$. Then $G_\xi H/H \subseteq X$ is a flat of X . Choose a ϑ -stable torus $T \subseteq G_\xi$. Then the flat is isomorphic to $A_X = T/(T \cap H)$. The space $\mathfrak{a} = \mathfrak{g}_\xi \cap \mathfrak{p} \cong \text{Lie } A_X$ is called a Cartan subspace of \mathfrak{p} . It is known that every semisimple element of \mathfrak{p} is conjugated by H to an element of \mathfrak{a} . This shows that all flats in X are conjugate to each other. Furthermore, W_X coincides with $N_G(T)/C_G(T)$, the little Weyl group of X (see e.g. [Kn1] p. 17/18). This shows that the W_X -action on the flat is induced by $N_G(T)$. These are the facts which make symmetric varieties so accessible. In general, neither are the flats conjugated to each other, nor is the W_X -action on them induced by elements of G .

7. Central valuations

Let \bar{X} be a normal embedding of X . The last step is to identify the closure $\bar{A}_{\bar{X}}$ of the generic flat. I can do that only in special cases. We recall some notion of [Kn4]. Let X be a G -variety and $K = k(X)$.

Definition: A valuation v of $k(X)$ with values in \mathbb{Q} is called *central* if it is G -invariant and vanishes on all non-zero elements of K^B . Let $\mathcal{Z}(X)$ be the set of central valuations.

For spherical varieties K^B equals k . Hence in this case all \mathbb{Q} -valuations are central. There is a short exact sequence

$$1 \longrightarrow (K^B)^\times \longrightarrow K^{(B)} \xrightarrow{f \mapsto \chi_f} \mathcal{X}(A_X) \longrightarrow 1.$$

By [Kn4] 3.6 every G -invariant valuation is uniquely determined by its restriction to $K^{(B)}$. The restriction of a central valuation factors through $\mathcal{X}(A_X)$. Hence, there is an injection $\mathcal{Z}(X) \hookrightarrow \mathcal{H}(X) := \text{Hom}(\mathcal{X}(A_X), \mathbb{Q})$. Usually, we will identify $\mathcal{Z}(X)$ with its image. Note that $\mathcal{H}(X)$ is a \mathbb{Q} -vector space of dimension $\text{rk } X$.

7.1. Theorem. ([Kn4] 6.5) *The subset $\mathcal{Z}(X) \subseteq \mathcal{H}(X)$ is a finitely generated convex cone with non-empty interior.*

The geometric meaning of central valuations is expressed in the next theorem.

7.2. Theorem. *Let X be a normal G -variety.*

1. *The valuation v_Z induced by a G -stable prime divisor $Z \subset X$ is central if and only if X and Z have the same complexity ([Kn4] 7.3).*

2. Let v be a non-trivial central valuation of X . Then there exists a smooth G -variety which is G -birational to X together with a G -stable prime divisor Z such that v_Z is a multiple of v ([Kn4] 4.4 and 7.2).

Consider $A = A_X$ as an A -variety by left translation. Then we obviously have $\mathcal{H}(A) = \mathcal{H}(X)$. Recall the classification of normal torus embeddings ([TE], [O]): The normal affine equivariant embeddings of A are classified by finitely generated strictly convex subcones $\mathcal{C} \subseteq \mathcal{H}(A)$. Let $\bar{A}(\mathcal{C})$ be the embedding determined by the cone \mathcal{C} . The orbits of $\bar{A}(\mathcal{C})$ correspond to the faces of \mathcal{C} . At the moment, we are only interested in the case where $\mathcal{C} = \mathbb{Q}^+v$ is a ray. Hence, $\bar{A}(v) = \bar{A}(\mathbb{Q}^+v)$ has two orbits: The open one and one of codimension one. Because all rays are admitted we have $\mathcal{Z}(A) = \mathcal{H}(X)$.

7.3. Theorem. *Let $X \hookrightarrow \bar{X}$ a normal equivariant embedding and $Z \subseteq \bar{X}$ a prime divisor such that $v = v_Z$ is central. Then for all $w \in W_X$ and generic $\hat{\alpha} \in \hat{T}_{\bar{X}}$ there is a (unique) open A_X -embedding $\bar{A}(wv) \hookrightarrow \bar{A}_{\bar{X}} = \bar{F}_{\hat{\alpha}}$ such that the closed orbit is mapped into Z .*

Proof: Because W_X acts on $\bar{A}_{\bar{X}}$ we may assume $w = 1$. By shrinking \bar{X} and X we may also assume that \bar{X} is smooth and that $Z = \bar{X} \setminus X$.

Choose a transcendence basis $f_j \in k(\tilde{X})^{(B)}$ of $k(\tilde{X})^U$ and let $\hat{\psi} : \tilde{X} \setminus D \times \mathbf{A}_r^s \hookrightarrow \hat{C} \subseteq \hat{T}_{\tilde{X}}$ be the open embedding as in section 4. Because $G \cdot \hat{C}$ is dense in $\hat{T}_{\tilde{X}}$ it suffices to consider flats $F_{\hat{\alpha}}$ with $\hat{\alpha}$ in the image of $\hat{\psi}$. We also fix $\underline{b} \in \mathbf{A}_r^s$.

The group G contains the structure group H of $\tilde{X} \rightarrow X$. Hence every B -divisor of \tilde{X} is the preimage of a unique B -divisor of \bar{X} which does not contain Z in its support. Let $\bar{D}(\underline{b})$ be the B -divisor corresponding this way to $D(\underline{b}) = \sum_j b_j [f_j]$. Let $D \subseteq \tilde{X}$ be the union of the supports of the $[f_j]$ and let $\bar{D} := \overline{p(D)} \subseteq \bar{X}$. Then we get the following commutative diagram:

$$\begin{array}{ccc} \tilde{X} \setminus D & \xrightarrow{\hat{\psi}(\cdot, \underline{b})} & \hat{T}_{\tilde{X}} \\ p \downarrow & & \downarrow \hat{\Phi} \\ \bar{X} \setminus \bar{D} & \xrightarrow{\psi_{\bar{D}(\underline{b})}} & \mathfrak{g}^* \end{array}$$

Observe, that \bar{D} does not contain Z , i.e., Z intersects $\bar{X} \setminus \bar{D}$ non-trivially.

Fix $x_0 \in \tilde{X} \setminus D$ and let gL be the image of $\hat{\psi}(x_0, \underline{b})$ in G/L . Then $L(\underline{b}) := gLg^{-1}$ is the isotropy group of $\psi_{\bar{D}(\underline{b})}(p(x_0))$ which is equipped with a homomorphism $L(\underline{b}) \rightarrow A_X$. Let $\Sigma(\underline{b})$ be the fiber of $\psi_{\bar{D}(\underline{b})}$ through $p(x_0)$. Then, by the local structure theorem, this induces an isomorphism

$$P(X) \times^{L(\underline{b})} \Sigma(\underline{b}) \xrightarrow{\sim} \bar{X} \setminus \bar{D}$$

The group $L(\underline{b})$ acts on $\Sigma(\underline{b})$ only via its quotient $A = A_X$ and the orbits in $X \cap \Sigma(\underline{b})$ are just twisted flats. Hence, it suffices to prove the theorem for the A -orbit closures in $\Sigma(\underline{b})$.

Because A is a torus there is an A -stable affine subset $\Sigma_0 \subseteq \Sigma(\underline{b})$ such that $Z_0 := Z \cap \Sigma_0$ is non-empty and all orbits in Z_0 are closed (see e.g. [Kn4] 2.10). Consider the categorical quotient $q : \Sigma_0 \dashrightarrow \Sigma_0 // A$. Now we use the main assumption, namely that v_Z is central, i.e., it vanishes on all B -invariant rational functions. Because $k(X)^B = k(\Sigma_0)^A$, this implies that $Z_0 \rightarrow \Sigma_0 // A$ is surjective and $k(\Sigma_0)^A = k(Z_0)^A = \text{Quot } k[\Sigma_0 // A]$. Hence the generic fiber of q contains a dense orbit. This implies that the generic fiber is an affine A -embedding with a closed orbit of codimension one, namely the intersection with Z . Therefore, the induced valuation corresponds to $v = v_Z$. This shows that the generic orbit closure in $\Sigma(\underline{b})$ contains $\bar{A}(v)$ such that the closed orbit is in Z . \square

Now we are in the position to prove the main result of this paper.

7.4. Theorem. *The action of the little Weyl group W_X on $\mathcal{H}(X) = \text{Hom}(\mathcal{X}(A_X), \mathbb{Q})$ is generated by reflections and $\mathcal{Z}(X)$ is one of its Weyl chambers. In particular, $\mathcal{Z}(X)$ is a simplicial cone and a fundamental domain of W_X .*

Proof: We know that $\mathcal{Z}(X)$ is a finitely generated cone with non-empty interior. Thus it suffices to show two things:

A: The reflections at faces of codimension one of $\mathcal{Z}(X)$ are contained in W_X .

B: Each W_X -orbit intersects $\mathcal{Z}(X)$ in at most one point.

A: This is a consequence of functorial properties. By [Kn4] 8.1.4, we have $\mathcal{Z}(\tilde{X}) = \mathcal{Z}(X)$. By replacing X with \tilde{X} we may assume that X is non-degenerate. Choose $v \in \mathcal{Z}(X)$ in the relative interior of a face of codimension one. Let \bar{X} be a smooth G -variety, G -birational to X , which contains a prime divisor Z such that the valuation v_Z is a multiple of v . Let $N = N_{\bar{X}}(Z)$ be its normal bundle. Then $P(X) = P(Z) \supseteq P(N)$ (by e.g. [Kn4] 2.13, 2.6). By [Kn4] 7.4 the equality $\mathcal{H}(N) = \mathcal{H}(X)$ holds. This implies $P(N) = P(X)$ and that N is non-degenerate. By the same theorem, $\mathcal{Z}(N)$ is the cone generated by $\mathcal{Z}(X)$ and $-v$. Hence $\mathcal{Z}(N)$ is a half space whose boundary \mathcal{B} is the hyperplane spanned by the chosen face. From $\mathcal{Z}(N) \neq \mathcal{H}(X)$ follows $N \neq G \cdot N^U$ ([Kn4] 8.5). This implies that W_N is non-trivial ([Kn1] 9.1).

By [Kn4] 8.2 there is a torus $H \subseteq \text{Aut}^G N$ with $k(N)^B \subseteq k(N)^H$ and such that $\mathcal{B} = \text{Hom}(\mathcal{X}(H), \mathbb{Q})$. Then Theorem 5.1 implies that W_N fixes \mathcal{B} pointwise. Hence W_N consists of the identity and the reflection at \mathcal{B} .

There is a flat deformation $Y \rightarrow \mathbf{A}^1$ such that the generic fiber is \bar{X} and the zero-fiber is N (see e.g. [Fu] §5.1). Then we have $W_N \subseteq W_Y$ and $W_Y = W_X$ by [Kn1] 6.5, 2. and 4. respectively. This shows the claim.

B: Assume, there is $v \neq w \cdot v \in \mathcal{Z}(X)$, $w \in W_X$. Let \bar{X} be G -birational to X containing two disjoint invariant divisors Z_1 and Z_2 whose valuations are proportional to v and wv , respectively. Then the closure $\bar{A}_{\bar{X}}$ of a generic twisted flat contains $\bar{A}(wv)$ whose closed

orbit is both in Z_1 and in Z_2 , a contradiction. \square

In [Kn1] a little Weyl group has been defined for arbitrary G -varieties.

7.5. Corollary. *The definition of W_X is independent of the choice of \tilde{X} and coincides with that of [Kn1].*

Proof: W_X is uniquely determined by $\mathcal{Z}(X)$ whose definition does not depend on the choice of \tilde{X} . This shows the first claim. For the second let W'_X be the little Weyl group as defined in [Kn1]. We know already $W'_X = W_{\tilde{X}} = W_X$ because \tilde{X} is non-degenerate (Lemma 3.4). Furthermore, $W'_X \subseteq W'_{\tilde{X}} = W_X$ by [Kn1] 6.5:1. Finally, the whole reasoning of part A of the proof above applies also to W'_X . The only thing one has to observe is that statement and proof of Theorem 5.1.3 are also valid without the non-degeneracy assumption for the Weyl groups as defined in [Kn1]. Hence, W'_X contains all reflections at faces of $\mathcal{Z}(X)$, which shows $W_X = W_{\tilde{X}} \subseteq W'_X$. \square

Here are some applications for an arbitrary normal G -variety X . Recall that the center of a valuation v is the largest closed subvariety Y with $v(\mathcal{O}_{X,Y}) \geq 0$.

Definition: A *source* of X is a non-empty subvariety $Y \subseteq X$ which is the center of a central valuation.

7.6. Corollary. *Let \bar{A}_X be the closure of a generic twisted flat and $v \in \mathcal{H}(X)$. Then there is a morphism of embeddings $\bar{A}(v) \rightarrow \bar{A}_X$ if and only if the unique valuation $wv \in \mathcal{Z}(X)$, for $w \in W_X$, has a non-empty center in X . In this case the closed orbit of $\bar{A}(v)$ is mapped into this center.*

Proof: We may assume $w = 1$. Let $X \hookrightarrow X_1$ be an equivariant completion (see [Su]). Then v has a non-empty center Y in X_1 . Choose a G -variety \bar{X} birational to X which contains a divisor Z with $v_Z = v$ and such that there is a morphism $\bar{X} \rightarrow X_1$. This morphism maps Z dominantly to Y . Hence, there is a morphism $\bar{A}(v) \subseteq \bar{A}_{\bar{X}} \rightarrow \bar{A}_X$ if and only if $Y \cap X \neq \emptyset$. \square

7.7. Corollary. *Each G -variety X contains only a finite number of sources and \bar{A}_X meets each one of them.*

Proof: The number of orbits in \bar{A}_X is finite. \square

7.8. Corollary. *1. The closure of a generic twisted flat is complete if and only if every valuation $v \in \mathcal{Z}(X)$ has a non-empty center in X .*

2. The generic twisted flat is closed if and only if X contains no proper source.

The next result has been proved by Brion for spherical varieties with an entirely different method (unpublished).

7.9. Corollary. *Let X be a normal affine variety, containing a proper source. Then there exist a non-trivial G -stable non-negative grading of $k[X]$.*

Proof: The embedding \bar{A}_X is affine. Hence its normalization is given by a cone $\mathcal{C} \subseteq \mathcal{H}(X)$. This cone must be W_X -stable. Because \bar{A}_X meets Y it cannot be trivial. Hence, there is $v \neq 0$ in the relative interior of \mathcal{C} . But then $v_0 := \sum_{w \in W_X} wv$ is W_X -invariant and also in the interior of \mathcal{C} , hence non-zero and contained in $\mathcal{Z}(X) \cap -\mathcal{Z}(X)$. To this v_0 corresponds a one parameter subgroup $H \subseteq \text{Aut}^G X$ by [Kn4] 8.2, which induces an invariant grading. The generic orbit of H is non-closed in the closure of a generic flat, hence in X , i.e., the grading is non-negative. \square

Remark: The corollary is most useful in the spherical case: A normal affine non-homogeneous spherical variety is graded.

8. Toroidal sources

The reasoning in the preceding section allows various generalizations which we didn't include there in order to prove Theorem 7.4 as soon as possible. Let $Y \subseteq X$ be a G -stable subvariety. Then we denote by $\mathcal{F}(Y)$ the set of all prime divisors $D \subseteq X$ with $Y \subseteq D$, $B \cdot D = D$ but $G \cdot D \neq D$. See [Kn4] for the relevance of this notion.

8.1. Theorem. *Let $X \hookrightarrow \bar{X}$ be an embedding and $Y \subseteq \bar{X}$ a source with $\mathcal{F}(Y) = \emptyset$. Then there is a $P(X)$ -stable open subset $X_0 \subseteq \bar{X}$ with $X_0 \cap Y \neq \emptyset$ and*

- a) *The quotient $\Sigma_0 := X_0/P_u(X)$ exists and L acts on it only via A_X .*
- b) *For every $\underline{b} \in \mathbf{A}_r^s$ there is an isomorphism $P(X) \times^L \Sigma_0 \xrightarrow{\sim} X_0$ such that the generic A_X -orbits are mapped to twisted flats.*
- c) *Conversely, there is a non-empty open subset V of $\widehat{T}_{\bar{X}}$ with: For every $\hat{\alpha} \in V$ there is a \underline{b} such that the twisted flat $F_{\hat{\alpha}}$ is the image of an A_X -orbit as in b).*

Proof: Same as the first part of the proof of Theorem 7.3, together with the following remark: Let $\bar{D}'(\bar{b})$ be the B -divisor of \bar{X} which one obtains by omitting all G -stable components and let \bar{D}' its support. Then by assumption $X_0 := \bar{X} \setminus \bar{D}'$ will intersect Y non-trivially and $\psi_{\bar{D}'(\bar{b})}$ differs from $\psi_{\bar{D}(\bar{b})}$ only by translation with a character of \mathfrak{g} . \square

With this theorem we are left with the study of the A_X -variety Σ_0 . The best thing which can happen is when Σ_0 is isomorphic to $\bar{A} \times V$ with some torus embedding \bar{A} . Therefore, we define:

Definition: A source Y in a normal G -variety X is called *toroidal* if $\mathcal{F}(Y) = \emptyset$ and each G -stable prime divisor of X containing Y is central. Let $\mathcal{V}_X(Y) \subseteq \mathcal{Z}(X)$ be the set of valuations attached to these divisors.

Let $Y \subseteq X$ be a toroidal source. Then the local ring $\mathcal{O}_{X,Y}$ is uniquely determined by $\mathcal{V}_X(Y)$ ([Kn4] 3.8).

8.2. Theorem. *Let $X \hookrightarrow \bar{X}$ be a normal embedding and let $Y \subseteq \bar{X}$ be a toroidal source.*

- a) *There is a normal affine A_X -embedding \bar{A} with closed orbit Z such that $\mathcal{V}_{\bar{A}}(Z) = \mathcal{V}_{\bar{X}}(Y)$ and which is contained in the closure of the generic flat.*
- b) *The closure $\bar{F}_{\hat{\alpha}}$ of a generic flat is transversal to Y , i.e., there is a smooth variety V , a point $v \in V$ and an open embedding $\bar{A} \times V \hookrightarrow \bar{X}$ such that $\bar{A} \times \{v\} \hookrightarrow \bar{F}_{\hat{\alpha}}$ and such that $Z \times V$ maps onto an open subset of Y .*

Proof: By Theorem 8.1 it suffices to study $Z_0 := (Y \cap X_0)/P_u(X)$ inside Σ_0 . The proof is very similar to the second part of that of Theorem 7.3. Again, we may assume that Σ_0 is affine and that Z_0 consists of closed orbits. Because every B -stable divisor containing Y is central, it follows that every A_X -invariant rational function is defined in Z_0 . This implies that the generic fiber of $q : \Sigma_0 \rightarrow \Sigma_0//A_X$ contains a dense orbit. Because principal bundles for a torus are locally trivial in the Zariski topology, there is a rational section σ of $\Sigma_0 \rightarrow \Sigma_0//A_X$, such that the image of σ is contained in the union of the open orbits. Then by Lemma 6.1 we may shrink Σ_0 such that it is isomorphic to $\bar{A} \times V$ where \bar{A} is the generic fiber and $V = \Sigma_0//A_X$. From this all assertions follow. \square

Now, we can use the theory of torus embeddings to conclude: Let $Y \subseteq X$ be a toroidal source. Then $\mathcal{V}_X(Y)$ is finite. Let $\mathcal{C}_X(Y) \subseteq \mathcal{H}(X)$ be the convex cone generated by $\mathcal{V}_X(Y)$. Then the elements of $\mathcal{V}_X(Y)$ span exactly the extremal rays of $\mathcal{C}_X(Y)$. In particular, $\mathcal{O}_{X,Y}$ is uniquely determined also by $\mathcal{C}_X(Y)$. It is easy to see that if X is quasihomogeneous then a source Y is toroidal if and only if $\mathcal{F}(Y) = \emptyset$. The reason is that Y is contained under these assumptions in only a finite number of B -stable prime divisors.

8.3. Corollary. *Let $X \hookrightarrow \bar{X}$ be a normal equivariant embedding such that all sources are toroidal. Let \mathfrak{F} be the set of $\mathcal{C}_{\bar{X}}(Y) \subseteq \mathcal{H}(X)$ where Y runs through all sources of \bar{X} . Then the closure of a generic flat is normal and is given as an A_X -embedding by the fan $W_X \mathfrak{F}$.*

Next I want to improve Corollary 6.2. Remember the map $\hat{T}_{\bar{X}} \rightarrow \mathfrak{a}_{\bar{X}}^*$. Let $\mathfrak{a}_{\bar{X}}^{rr} \subseteq \mathfrak{a}_{\bar{X}}^r$ be the largest open subset over which all fibers are irreducible. This set is not empty (see proof of Lemma 3.4).

8.4. Theorem. *Let $X \subseteq \bar{X}$ be a normal embedding and assume that for every source Y in \bar{X} the set $\mathcal{F}(Y)$ is empty. Then an open subset $S' \subseteq \hat{T}_{\bar{X}}$ can be found having the properties of Corollary 6.2, and additionally that the image of $S' \rightarrow \mathfrak{a}_{\bar{X}}^r$ contains $\mathfrak{a}_{\bar{X}}^{rr}$.*

Proof: Let $Y \subseteq \bar{X}$ be a source and X_0 as in Theorem 8.1. Let \bar{A} be the closure of a generic A_X -orbit in $X_0/P_u(X)$. It follows from that theorem that there is an open subset $S(Y) \subseteq \hat{T}_{\bar{X}}$ with $S(Y) \rightarrow \mathfrak{a}_{\bar{X}}^r$ such that $\bar{A} \hookrightarrow \bar{F}_{\hat{\alpha}}$ for all $\hat{\alpha} \in S(Y)$. Now set

$S' := \bigcap_{w,Y} wS(Y)$ where w runs through W_X and Y through all sources. Then S' has all required properties because the intersection of finitely many non-empty open subsets is non-empty on an irreducible variety. \square

In general, the set S' will depend on the embedding \bar{X} and no flat may be good for all \bar{X} . But let X be a homogeneous spherical variety with an embedding $X \hookrightarrow \bar{X}$. Then every valuation is central, hence every orbit of \bar{X} is a source, and the condition of Theorem 8.4 means that no B -stable divisor of X contains a G -orbit in its closure. These embeddings are called toroidal (comp. [BL] 2.1). Now we get really specific flats with a good behavior, independently of \bar{X} :

8.5. Corollary. *Let $X = G/H$ be spherical and \bar{X} a toroidal embedding. Let $S' \subseteq \widehat{T}_{\bar{X}}$ be the preimage of $\mathfrak{a}_{\bar{X}}^{rr}$ and let $\bar{A}_{\bar{X}}$ be the closure of a generic twisted flat. Then the morphism $S' \times A_X \rightarrow S' \times X : (\hat{\alpha}, a) \mapsto (\hat{\alpha}, \zeta_{\hat{\alpha}}(a))$ extends to a closed embedding $S' \times \bar{A}_{\bar{X}} \rightarrow S' \times \bar{X}$.*

Proof: Because \tilde{X} is also spherical, we can identify the parameter space \mathbf{A}_r^s with $\mathfrak{a}_{\tilde{X}}^*$. Then the existence of the open embedding

$$G \times^{P(X)} [\tilde{X} \setminus D \times \mathfrak{a}_{\tilde{X}}^r] \hookrightarrow \widehat{T}_{\tilde{X}}$$

shows that each fiber of $\widehat{\Psi} : \widehat{T}_{\tilde{X}} \rightarrow \mathfrak{a}_{\tilde{X}}^r$ contains an open G -orbit. This implies that each fiber F of $\widehat{\Phi} : S' \rightarrow G/L \times \mathfrak{a}_{\tilde{X}}^{rr}$ contains an open, hence dense A_X -orbit. The image of this dense orbit in \tilde{X} is a flat. Because \tilde{X} is homogeneous, this subspace is closed (Corollary 7.8). Hence all fibers F are homogeneous. Then the assertion follows from Theorem 8.4. \square

9. Logarithmic and twisted cotangent bundles

In this section we discuss some variants of the cotangent bundle. Let me assume that $X = \tilde{X}$ is non-degenerate. Flats are the images in X of the A_X -orbits in \widehat{T}_X and we studied their closure in certain embeddings \bar{X} . It is convenient to have the same closure for the orbits themselves. This is possible by extending the bundles T_X^* and \widehat{T}_X to all of \bar{X} . Just taking $T_{\bar{X}}^*$ doesn't work, one reason being, that it is not a vector bundle if \bar{X} is not smooth.

Assume again that every source of \bar{X} is toroidal. Let $D \subset \bar{X}$ be the union of all central divisors. Then let $\Omega_{\bar{X}}[D]$ be the sheaf of differential forms with logarithmic singularities along D , i.e., it is locally generated by $\Omega_{\bar{X}}$ together with sections of the form $f^{-1}df$ where f is a rational function having its poles and zeros in D (see [O] 3.1). Let $T_{\bar{X}}^*[D] = \text{Spec}_{\bar{X}} S^* \Omega_{\bar{X}}[D]^\vee$ be the geometric realization of $\Omega_{\bar{X}}[D]$.

9.1. Theorem. *Let X be non-degenerate and $X \hookrightarrow \bar{X}$ an equivariant normal embedding such that every source is toroidal.*

- a) The moment map $T_X^* \rightarrow \mathfrak{g}^*$ extends to $T_X^*[D] \rightarrow \mathfrak{g}^*$. In particular, $\widehat{T}_{\overline{X}}[D]$ is defined as an irreducible component of $T_{\overline{X}}^*[D] \times_{\mathfrak{t}^*/W} \mathfrak{a}_X^r$.
- b) There is a non-empty G -stable open subset $\overline{X}_0 \subseteq \overline{X}$, meeting every source, such that $T_{\overline{X}}^*[D]$ is a vector bundle over \overline{X}_0 .
- c) For $\widehat{\alpha} \in \widehat{T}_X$ generic, let \overline{A} be the closure of the orbit $A_X \widehat{\alpha}$ in $\widehat{T}_{\overline{X}}[D]$. Then $\widehat{\pi} : \overline{A} \rightarrow \overline{X}$ is a closed embedding, hence an isomorphism onto a flat.

Proof: a) Let $\xi \in \mathfrak{g}$ and $f \in \mathcal{O}_{\overline{X}}$ be an invertible function on some open subset $V \subseteq X$. Because D is G -stable we have $v_{D_0}(\xi_* f) \geq v_{D_0}(f)$ for every component of D . Hence, $f^{-1}df(\xi_*) = f^{-1}\xi_* f$ regular along D , i.e., the moment map extends.

b) Let $Y \subseteq \overline{X}$ be a source and let $\overline{A} := \overline{A}(\mathcal{C}(Y))$. Then there is a smooth variety V such that $V \times \overline{A}$ is isomorphic to an open subset of \overline{X} which intersects Y non-trivially (Theorem 8.2). Let $D_0 := \overline{A} \setminus A_X$ the boundary. Then $V \times D_0$ corresponds to D . This reduces the assertion to the case of the torus embedding \overline{A} . Then $\Omega_{\overline{X}}[D_0]$ is spanned by $\chi^{-1}d\chi$ where χ is a character of A_X ([O] Prop. 3.1). This means that $T_{\overline{A}}^*[D_0] = \overline{A} \times \mathfrak{a}_X^*$ is just the trivial bundle, in particular a vector bundle.

c) By the same reasoning it suffices to study \overline{A} . Then $\widehat{T}_{\overline{A}}[D_0] = T_{\overline{A}}^*[D_0] = \overline{A} \times \mathfrak{a}_X^*$ proves the claim. \square

The most important application is again for spherical varieties. A spherical variety is called toroidal if all orbits (=sources) are toroidal. These are classified by fans \mathfrak{F} which are supported in $\mathcal{Z}(X)$ (see [Kn3] for details). Let $W_X \mathfrak{F}$ be the fan consisting of all cones $w\mathcal{C}$ where $w \in W_X$ and $\mathcal{C} \in \mathfrak{F}$.

9.2. Corollary. *Let X be a non-degenerate toroidal spherical variety corresponding to a fan \mathfrak{F} . Consider the moment map on the logarithmic cotangent bundle $T_X^*[D] \rightarrow \mathfrak{g}^*$. Then the fiber over any element of \mathfrak{a}^{rr} is the disjoint union of varieties isomorphic to the A_X -embedding corresponding to the fan $W_X \mathfrak{F}$.*

Proof: It suffices to prove that for $\widehat{\Phi} : \widehat{T}_X[D] \rightarrow G/L \times \mathfrak{a}^*$. The proof of Corollary 8.5 showed that every fiber over $G/L \times \mathfrak{a}^{rr}$ is irreducible with dense A_X -orbit. Then Corollary 8.3 and Theorem 9.1 imply that the fiber is the torus embedding attached to $W_X \mathfrak{F}$. \square

That the fibers are in general not connected is due to the fact that W_X may not equal the normalizer $W(\mathfrak{a}^*)$. The number of components is always equal to the index of W_X inside $W(\mathfrak{a}^*)$. The corollary is a weak version for spherical varieties of a theorem of Abeasis [Ab]. There, all fibers of the moment map for symmetric varieties over semisimple points are described.

Finally let me mention a slight reformulation of the theory. To make the theory work also for degenerate varieties, we applied our theorems to the cotangent bundle of an

auxiliary space \tilde{X} . With a reformulation it is possible to do this the other way round, namely to keep X but to replace T_X^* by another bundle namely a twisted (or shifted) cotangent bundle introduced by Ginzburg and Kostant (see [BB] section 2 for details). This has the advantage that the group H disappears (almost). Remember $H \subseteq G$ by definition and the short exact sequence

$$0 \longrightarrow \mathfrak{a}_X^* \longrightarrow \mathfrak{a}_{\tilde{X}}^* \longrightarrow \mathfrak{h}^* \longrightarrow 0.$$

Hence we have the diagram

$$\begin{array}{ccc} \widehat{T}_{\tilde{X}} & \longrightarrow & T_{\tilde{X}}^* \\ \downarrow & & \downarrow q \\ \mathfrak{a}_{\tilde{X}}^* & \longrightarrow & \mathfrak{h}^* \end{array}$$

Both vertical arrows are H -invariant and H acts freely on the fibers (because it acts already freely on \tilde{X}). Choose $\lambda \in \mathfrak{h}^*$. Then one defines the *twisted cotangent bundle* as $T_X^\lambda := q^{-1}(\lambda)/H$ and analogously its polarized form \widehat{T}_X^λ . It is easily verified that $T_X^\lambda \rightarrow X$ is a bundle which is locally isomorphic to T_X^* . Furthermore, the symplectic structure of $T_{\tilde{X}}^*$ induces one on T_X^λ (Hamiltonian reduction). Note, that G acts on T_X^λ via its quotient $\overline{G} = G/H$ (the group we started with). Similarly, on \widehat{T}_X^λ only $A_X = A_{\tilde{X}}/H$ is acting. We call the image in X of an A_X -orbit in \widehat{T}_X^λ a *flat with twist* λ .

The moment map on $T_{\tilde{X}}^*$ descends to a moment map $T_X^\lambda \rightarrow \overline{\mathfrak{g}}^*$. We call T_X^λ *non-degenerate* if λ is contained in the image of $\mathfrak{a}_{\tilde{X}}^{rr} \rightarrow \mathfrak{h}^*$. Then Theorem 8.4 implies

9.3. Theorem. *Let T_X^λ be non-degenerate and $X \hookrightarrow \overline{X}$ an embedding. Then the closure in \overline{X} of a generic flat with twist λ is W_X -symmetric and contains $\overline{A}(w\mathcal{C}(Y))$ for all $w \in W_X$ and every toroidal source $Y \subseteq \overline{X}$.*

Finally, there is also a logarithmic-twisted version of this theorem, whose statement and proof I leave to the reader.

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