Some remarks on multiplicity free spaces [*]

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Abstract

We study multiplicity free representations of connected reductive groups. First we give a simple criterion to decide the multiplicity freeness of a representation. Then we determine all invariant differential operators in terms of a finite reflection group, the little Weyl group, and give a characterization of the spectrum of the Capelli operators. At the end, we reproduce the classification of multiplicity free representations (without proof) annotated with some basic data.

1 Introduction

A finite-dimensional representation V of a connected reductive group G is called multiplicity free if its coordinate ring contains every simple G-module at most once. Our main results are: we give a simple criterion to decide the multiplicity freeness of a representation (Theorem 3.3), we determine all invariant differential operators in terms of a finite reflection group, the little Weyl group (Theorem 4.8), and give a characterization of the spectrum of the Capelli operators (Theorem 4.10). At the end, we reproduce the classification of multiplicity free representations (obtained by Kac, Brion, Benson-Ratcliff, Leahy) annotated with same basic data.

Multiplicity free representations form a very restricted class of representations. Nevertheless they are very important due to Roger Howe's philosophy that every "nice" result in the invariant theory of particular representations can be traced back to a multiplicity free representation. This holds for example for the Capelli identities [HoUm]. Also all of Weyl's first and second fundamental theorems can be explained by some multiplicity freeness result.

None of the results in this paper is really new. The multiplicity freeness criterion is a simple corollary of the local structure theorem of Brion-Luna-Vust [BLV]. The determination of invariant differential operators is a very special case of a much more general result valid

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for any G-variety [Knop2]. In our situation the proof simplifies tremendously. In fact, except for the table at the end, this paper is completely self-contained. The characterization of the spectrum of Capelli operators then follows the lines of [Sahi], where it was proved in special cases. Finally, the classification is added just for convenience. New is the calculation of the Weyl groups.

2 The local structure theorem

In this section we present the local structure theorem of Brion-Luna-Vust [BLV] in the form of [Knop1].

Let G be a connected reductive group and X any affine G-variety. For a function f on X let X_f be the set of points where f is non-zero. The Lie algebra $\mathfrak{g} := \operatorname{Lie} G$ acts on functions by derivations, hence we can consider the morphism

$$\psi_f : X_f \to \mathfrak{g}^* : x \mapsto [\xi \mapsto \frac{\xi f(x)}{f(x)}].$$

Let $B \subseteq G$ be a Borel subgroup and let $f \in \mathbb{C}[X]$ be a highest weight vector with respect to B. Then $P_f := \{g \in G \mid gf \in \mathbb{C}^* f\}$ is a parabolic subgroup containing B having a character χ_f with $gf = \chi_f(g)f$ for all $g \in P_f$. Let P_f^u be its unipotent radical and $\mathfrak{p}_f$, $\mathfrak{p}_f^u$ the respective Lie algebras.

2.1 Lemma *The roots of P_f are exactly those roots α for which $\langle \chi_f | \alpha^\vee \rangle \geq 0$.*

Proof This is a well known property of highest weight vectors. ☐

Occasionally, we will identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using a G-invariant scalar product $(\cdot, \cdot)$. Let $T \subseteq B$ be a maximal torus with Lie algebra $\mathfrak{t}$. Then for a character $\chi \in \mathfrak{t}^*$ let χ' be the corresponding element in $\mathfrak{t}$, i.e., $(\chi', \xi) = \chi(\xi)$ for all $\xi \in \mathfrak{t}$.

2.2 Theorem *Let $f \in \mathbb{C}[X]$ be a highest weight vector. Then the morphism $\psi_f : X_f \to \mathfrak{g}^*$ is P_f-equivariant, its image is a single P_f-orbit, namely $\chi'_f + \mathfrak{p}_f^u$, and every isotropy group of this orbit is a Levi complement of P_f.*

Proof For $p \in P_f$ and $x \in X_F$ we have

$$\begin{aligned} \psi_f(p^{-1}x)(\xi) &= \frac{\xi f(p^{-1}x)}{f(p^{-1}x)} = \frac{{}^p(\xi f)(x)}{{}^pf(x)} = \frac{{}^p\xi {}^pf(x)}{{}^pf(x)} = \frac{\chi_f(p){}^p\xi f(x)}{\chi_f(p)f(x)} \\ &= \psi_f(x)({}^p\xi)) = {}^{p^{-1}}\psi_f(x)(\xi). \end{aligned}$$

Thus, ψ_f is P_f-equivariant. For $\xi \in \mathfrak{p}_f$ we have

$$\psi_f(x)(\xi) = \frac{\xi f(x)}{f(x)} = \frac{\chi_f(\xi)f(x)}{f(x)} = \chi_f(\xi).$$

This shows that, using the identification $\mathfrak{g}^* = \mathfrak{g}$, we have $\psi_f(x) - \chi'_f \in \mathfrak{p}_f^\perp = \mathfrak{p}_f^u$. Thus, the image of ψ_f is contained in $\chi'_f + \mathfrak{p}_f^u$.

Claim: $\chi'_f + \mathfrak{p}_f^u$ is a single P_f-orbit. In fact, for every root α lying in $\mathfrak{p}_f^u$ we have $\langle \chi_f | \alpha^\vee \rangle \neq 0$ by Lemma 2.1. Hence also $\alpha(\chi'_f) \in \mathbb{C}^*(\chi'_f, \alpha^\vee) \neq 0$. This shows that the centralizer of χ'_f in P_f^u is trivial. Hence $P_f^u \chi'_f$ is an open orbit in $\chi'_f + \mathfrak{p}_f^u$. But, as an orbit of a unipotent group in an affine variety, it is also closed which proves the claim.

Finally, the isotropy group of χ'_f in P is a Levi complement of P_f, again by Lemma 2.1. This proves the last assertion. ☐

2.3 Corollary *For $x \in X_f$ let L be the isotropy group of $\psi_f(x)$ in P_f. Then L is a Levi complement of P_f and $\Sigma := \psi_f^{-1}(\psi_f(x))$ is an affine L-stable subvariety of X_f such that $P_f \times^L \Sigma \to X_f$ is a P_f-equivariant isomorphism.*

2.4 Theorem *Notation as above. Let f be such that P_f is of minimal dimension. Then the commutator subgroup of L acts trivially on Σ.*

Proof If (L, L) does not act trivially on Σ there is a highest weight vector $f' \in \mathbb{C}[\Sigma]$ for L which is not fixed by $L' := (L, L)$. Then f' extends uniquely to a B-semiinvariant function, also denoted f', of X_f. For $N \gg 0$, the function $h := f^N f'$ is regular. Moreover, its weight does not extend to a character of P_f. This shows $P_h \subsetneqq P_f$ contradicting the minimality of P_f. □

Remark Clearly, the parabolic subgroup P_f of minimal dimension depends only on X, and is denoted by $P(X)$.

3 Multiplicity free spaces

We apply the results of the preceding section to study a very restricted but nevertheless important class of varieties. Let V be a finite dimensional G-module. By abuse of language, V is called *multiplicity free* if its coordinate ring $\mathcal{P} := \mathbb{C}[V]$ contains every simple G-module at most once. A more geometric criterion is ([VinKim]):

3.1 Theorem *Let $B \subseteq G$ be a Borel subgroup. Then V is multiplicity free if and only if B has an open orbit in V.*

Proof Assume that $\mathcal{P}$ contains two different but isomorphic simple submodules M_1 and M_2. Let f_i be a highest weight vector of M_i. Then f_1 and f_2 are B-semiinvariant functions for the same weight of B. Thus $h := f_1/f_2$ is a non-constant B-invariant rational function on V making it impossible for B to have an open orbit.

Assume, conversely, that there is no open B-orbit, i.e., all orbits have positive codimension. Let f and Σ be as in Theorem 2.4. The action of L on Σ factors through the torus $A = L/(L, L)$. Moreover, Σ cannot admit an open orbit either (Corollary 2.3). Suppose all weight spaces of $\mathbb{C}[\Sigma]$ were one dimensional. Choose weight vectors $f_1, \ldots, f_r$ which generate $\mathbb{C}[\Sigma]$ as an algebra and choose $x_0 \in \Sigma$ with $f_i(x_0) \neq 0$ for all i. Every weight vector f is a monomial in the f_i, thus $f(x_0) \neq 0$. This means that the algebra homomorphism $\mathbb{C}[\Sigma] \to \mathbb{C}[A]$ corresponding to the orbit map $A \to \Sigma : a \mapsto ax_0$ is injective. Hence the orbit map is dominant, i.e., Ax_0 is dense. Contradiction.

We conclude that there are two non-proportional weight vectors f_1, f_2 with the same weight. These functions can be uniquely extended to B-semiinvariants on X_f which are also denoted by f_i. Then, for $N \gg 0$, we obtain two regular highest weight vectors $f^N f_1$ and $f^N f_2$ with the same weight. Thus the G-modules M_1 and M_2 generated by them are simple, different but isomorphic. □

Remark The second part is usually proved by using a non-trivial theorem of Rosenlicht (cf. [Kraft] II.4.3.E).

By definition, $\mathcal{P}$ has a decomposition $\mathcal{P} = \bigoplus_{\lambda\in\Lambda} \mathcal{P}_\lambda$, where Λ is a set of dominant weights and $\mathcal{P}_\lambda$ is a simple G-module with *lowest*[1] weight $-\lambda$.

3.2 Theorem *There are linear independent weights $\lambda_1, \ldots, \lambda_r$ (where r is called the* rank *of V) such that $\Lambda = \mathbb{N}\lambda_1 + \ldots + \mathbb{N}\lambda_r$. Let $f_i \in \mathcal{P}_{\lambda_i}$ be a highest weight vector. Then every highest weight vector of $\mathcal{P}$ is a scalar multiple of $f_1^{a_1} \ldots f_r^{a_r}$ for some $a_i \in \mathbb{N}$.*

Proof Let $f_\lambda \in \mathcal{P}_\lambda$ and $f_\mu \in \mathcal{P}_\mu$ be highest weight vectors. Then $f_\lambda f_\mu$ is non-zero hence a highest weight vector in $\mathcal{P}_{\lambda+\mu}$. This shows that Λ is additively closed. The rest is a consequence of the fact that $\mathcal{P}$ is factorial: let $\lambda_1, \lambda_2, \ldots \in \Lambda$ be the weights such that f_{λ_i} is an irreducible polynomial. For any highest weight vector $f_\lambda \in \mathcal{P}_\lambda$, let $f_\lambda = f_1 \ldots f_d$ be the prime factor decomposition. For every $b \in B$ we get another factorization $f \in \mathbb{C}^* \, {}^b f_\lambda = \mathbb{C}^* \, {}^b f_1 \ldots {}^b f_d$ which has to coincide with the previous one up to scalar multiples and the order of the factors. Since B is connected, the order of the factors is in fact preserved, thus all f_i are also B-semiinvariant. This shows that f_λ is a multiple of a monomial in the f_{λ_i} and $\Lambda = \sum_i \mathbb{N}\lambda_i$. Distinct monomials have different weights since otherwise V would not be multiplicity free. This shows that the sum is direct. □

Next we describe a simple algorithm for deciding whether a given representation is multiplicity free. For this, consider pairs (Δ, Ψ) where Δ is the set of positive roots of a Levi subgroup L of G and Ψ is the multiset (i.e., some elements may appear more than once) of weights of representation V of L. We define (Δ, Ψ) to be multiplicity free if V is a multiplicity free representation of L. For a highest weight $\chi \in \Psi$ (i.e., $\langle \chi, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$) let $S_\chi := \{\alpha \in \Delta \mid \langle \chi, \alpha^\vee \rangle > 0\}$. The following theorem allows one to recognize multiplicity freeness recursively:

3.3 Theorem

(a) *If $S_\chi = \emptyset$ for all highest weights $\chi \in \Psi$, then (Δ, Ψ) is multiplicity free if and only if Ψ is linearly independent.*

(b) *If there is a highest weight $\chi \in \Psi$ with $S_\chi \neq \emptyset$, then put $\Delta' := \Delta \backslash S_\chi$ and $\Psi' := \Psi \backslash \{\chi - \alpha \mid \alpha \in S_\chi\}$. Then (Δ, Ψ) is multiplicity free if and only if (Δ', Ψ') is multiplicity free.*

Proof In the first case, the commutator subgroup of L acts trivially on V. Then (a) follows from the fact that a torus has a dense orbit on a vector space if and only if the weights are linearly independent.

For (b) let $v_0 \in V$ be a highest weight vector with weight χ, and let $f \in V^*$ be a lowest weight vector of weight $-\chi$ with $f(v_0) = 1$. We want to apply Corollary 2.3 to $X = V$ and f (which is a highest weight vector with respect to the opposite Borel subgroup). Then Δ' is the set of positive roots of L, and $\mathfrak{p}_f^u$ corresponds exactly to S_χ. Define the L-module $N := \{v \in V \mid (\mathfrak{p}_f^u f)(v) = 0\}$. Since $\mathfrak{p}_f^u f$ is an L-module with weights $-\chi + S_\chi$, the set of weights of N is Ψ'.

The set Σ of Corollary 2.3 is the set of $v \in V$ with $f(v) \neq 0$ and $\xi f(v)/f(v) = \xi f(v_0)/f(v_0)$ for all $\xi \in \mathfrak{g}$. By the choice of f, the latter condition is trivially satisfied for $\xi \in \mathfrak{p}_f^-$, the parabolic opposite to $\mathfrak{p}_f$. For $\xi \in \mathfrak{p}_f^u$ it reads $\xi f(v) = 0$ (since $\xi f(v_0) = -f(\xi v_0) = 0$). We conclude that Σ is an open subset of N.

Finally, observe that B has an open orbit in V if and only if $L \cap B$ has an open orbit in Σ. Thus (b) follows from Theorem 3.1. □

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[1] This is more convenient for practical computations.

Remark This algorithm terminates with some final pair (Δ_0, Ψ_0) for which case (a) applies. Clearly, Δ_0 is just the set of positive roots of the Levi part of the parabolic subgroup P_f from Theorem 2.4. Therefore, Δ_0 is unique and allows one to determine $P(V)$. On the other hand, example 3 below shows that Ψ_0 is not unique. Nevertheless, the space spanned by Ψ_0 coincides with the space spanned by Λ. In particular, one can read off the rank of V. It is unclear how much further information can be extracted from Ψ_0. It would be nice to have an easy method to determine Λ. Finally, it should be remarked that in general this algorithm can be used to determine the so-called complexity of V. This is the minimal codimension of a B-orbit and equals $\#\Psi_0 - \dim\langle\Psi_0\rangle_{\mathbb{C}}$.

Examples 1 Let $G := GL(2, \mathbb{C})$, acting on $V := S^d(\mathbb{C}^2)$. Then $\Delta = \{\varepsilon_1 - \varepsilon_2\}$ and $\Psi = \{d\varepsilon_1, (d-1)\varepsilon_1 + \varepsilon_2, \ldots, d\varepsilon_2\}$. Thus $\chi = d\varepsilon_1$, $\Delta' = \emptyset$ and $\Psi' = \Psi \setminus \{\chi - (\varepsilon_1 - \varepsilon_2\} = \{d\varepsilon_1, (d-2)\varepsilon_1 + 2\varepsilon_2, \ldots, d\varepsilon_2\}$. This latter set is linearly independent if and only if $d \leq 2$. Thus among the binary forms, only $\mathbb{C}^2$ and $S^2(\mathbb{C}^2)$ are multiplicity free with rank 1 and 2, respectively.

2 Let $G := GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$, acting on $V := \mathbb{C}^m \otimes \mathbb{C}^n$. We may assume $m \leq n$. Then

$$\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq m\} \cup \{\varepsilon'_i - \varepsilon'_j \mid 1 \leq i < j \leq n\}$$

and

$$\Psi = \{\varepsilon_i + \varepsilon'_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Take $\chi = \varepsilon_1 + \varepsilon'_1$. Then

$$S_\chi = \{\varepsilon_1 - \varepsilon_j \mid 1 < j \leq m\} \cup \{\varepsilon'_1 - \varepsilon'_j \mid 1 < j \leq n\}.$$

Thus

$$\Delta' = \{\varepsilon_i - \varepsilon_j \mid 2 \leq i < j \leq m\} \cup \{\varepsilon'_i - \varepsilon'_j \mid 2 \leq i < j \leq n\}$$

and

$$\Psi' = \{\varepsilon_1 + \varepsilon'_1\} \cup \{\varepsilon_i + \varepsilon'_j \mid 2 \leq i \leq m, 2 \leq j \leq n\}.$$

Thus, we are in essentially the same situation as before with $L = GL_1(\mathbb{C}) \times GL_{m-1}(\mathbb{C}) \times GL_{n-1}(\mathbb{C})$ acting on $\mathbb{C}\chi \oplus \mathbb{C}^{m-1} \otimes \mathbb{C}^{n-1}$. This procedure stops when $m = 1$, leaving us with the linearly independent set of weights $\{\varepsilon_i + \varepsilon'_i \mid 1 \leq i \leq m\}$. Thus $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ acting on $V := \mathbb{C}^m \otimes \mathbb{C}^n$ is multiplicity free of rank $\min(m, n)$, a well-known fact.

3 In the last example, details are left to the reader. We take $G = Sp_{2n}(\mathbb{C}) \times GL_3(\mathbb{C})$, acting $\mathbb{C}^{2n} \otimes \mathbb{C}^3$. One applies the procedure three times with $\chi = \varepsilon_1 + \varepsilon'_1$, $\varepsilon_2 + \varepsilon'_2$, and $\varepsilon_3 + \varepsilon'_3$, respectively. After that one is left with three more weights: $-\varepsilon_1 + \varepsilon'_2$, $-\varepsilon_1 + \varepsilon'_3$, and $-\varepsilon_2 + \varepsilon'_3$. These six weights are linearly independent, therefore the representation multiplicity free of rank 6.

It should be noted that at the second step there are two different χ's to choose from: $\varepsilon_2 + \varepsilon'_2$ and $-\varepsilon_1 + \varepsilon'_2$. Choosing the second weight leads to a different set of linearly independent six weights. Thus the final set of weights is not unique.

If one takes $G' = Sp_{2n}(\mathbb{C}) \times SL_3(\mathbb{C})$ instead, then one ends up with the same set of weights but with the additional relation $\varepsilon'_1 + \varepsilon'_2 + \varepsilon'_3 = 0$. Thus, the six weights are then living in a five dimensional vector space, which implies that the representation is no longer multiplicity free.

4 Harmonic analysis on multiplicity free spaces

If V is multiplicity free then its dual space V^* is so, as well. In fact, its coordinate ring $\mathcal{D} := \mathbb{C}[V^*] = S^\bullet(V)$ decomposes as $\bigoplus_{\lambda\in\Lambda} \mathcal{D}_\lambda$, where $\mathcal{D}_\lambda$ is a simple G-module with *highest* weight λ. Thus, we have $\Lambda(V^*) = -w_0\Lambda(V)$, where w_0 is the longest element of the Weyl group.

The ring $\mathcal{D}$ can also be identified with the set of constant coefficient differential operators (whence the notation). In the sequel, we are going to denote the coordinate ring of $V \oplus V^*$ by $\mathcal{P} \otimes \mathcal{D}$, while $\mathcal{PD}$ is the algebra of linear differential operators on V. Clearly, multiplication furnishes an isomorphism of G-modules $m : \mathcal{P} \otimes \mathcal{D} \to \mathcal{PD}$.

Taking invariants, we conclude that both $(\mathcal{P} \otimes \mathcal{D})^G$ and $(\mathcal{PD})^G$ decompose as

$$\bigoplus_{\lambda,\mu\in\Lambda} (\mathcal{P}_\lambda \otimes \mathcal{D}_\mu)^G.$$

Since $\mathcal{D}_\mu$ is isomorphic to the dual representation $\mathcal{P}^*_\mu$, each summand is zero unless $\mu = \lambda$, in which case it is isomorphic to $\mathbb{C}$.

Definition The functions in $(\mathcal{P}_\lambda \otimes \mathcal{D}_\lambda)^G$ are called the *spherical functions* of weight λ. The elements of $m(\mathcal{P}_\lambda \otimes \mathcal{D}_\lambda)^G$ are called the *Capelli operators* of weight λ.

Clearly, the weight λ determines a spherical function, respectively a Capelli operator uniquely up to a scalar. In this paper we will make no attempt to normalize them. Instead we just pick for every weight one spherical function E_λ which in turn determines the Capelli operator $D_\lambda := m(E_\lambda)$. We can think of $V \oplus V^*$ as the cotangent bundle of V. Then E_λ is just the symbol of D_λ.

The terminology is explained as follows. Let $H \subseteq G$ (resp. $H^* \subseteq G$) be the isotropy group of the open G-orbit in V (resp. V^*). Then $V \otimes V^*$ contains $X := G/H \times G/H^*$ as open subset on which G acts diagonally. If $f(g_1H, g_2H^*)$ is a G-invariant on X, then $h(g) := f(H, gH^*)$ is a function on G which is constant on the double cosets for $H \times H^*$. Conversely, given h then $f(g_1H, g_2H^*) = h(g_1^{-1}g_2)$ is a G-invariant on X. Thus, we obtain an embedding of $(\mathcal{P} \otimes \mathcal{D})^G$ into ${}^H\mathbb{C}[G]^{H^*}$. The particular functions E_λ are characterized as being eigenfunctions of a commuting set of differential operators (Theorem 4.11). Thus our definition of a spherical function comes close to the classical notion of a spherical function.

Another way to see spherical functions is: let $K \subseteq G$ be a maximal subgroup. Let $V_\mathbb{R}$ equal V but considered as *real* vector space. Let $z_1, \ldots, z_n$ be coordinates on V. Then $\mathbb{C}[V_\mathbb{R}]$ is a polynomial ring in the $2n$ variables $z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n$ which can be identified with $\mathcal{P} \otimes \mathcal{D}$. Thus spherical functions can be thought of also as K-invariant polynomials on $V_\mathbb{R}$.

Consider the special case of $G = GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ acting on $V = \mathbb{C}^n \otimes \mathbb{C}^n$, i.e., $n \times n$-matrices by $(A_1, A_2)X = A_1XA_2^t$. Then V has coordinate functions $x_{ij} \in V^* \subseteq \mathcal{P}$ and corresponding partial derivatives $\partial_{ij} \in V \subseteq \mathcal{D}$. Moreover, one of the $\mathcal{P}_\lambda$'s is one-dimensional and spanned by the function $\det(x_{ij})$. Thus we get $D_\lambda = \det(x_{ij})\det(\partial_{ij})$ which is one side (usually the one on the right) of the Capelli identity. This explains why we call the D_λ Capelli operators. See [HoUm] for more on the connection between multiplicity free spaces and Capelli identities.

Remember that the monoid Λ is freely generated by weights $\lambda_1, \ldots, \lambda_r$ (Theorem 3.2). Let $f_i \in \mathcal{P}_{\lambda_i}$ be a highest weight vector (it has, by our convention, weight $-w_0\lambda_i$). Let $\mathfrak{a}^*$ be the $\mathbb{C}$-vector space spanned by Λ. We are going to link $(\mathcal{P} \otimes \mathcal{D})^G$ and $(\mathcal{PD})^G$ with $\mathfrak{a}^*$.

Let $V_0 \subseteq V$ be the open B-orbit. Then every highest weight vector $f \in \mathbb{C}[V]$ is invertible on V_0. Thus it gives rise to a B-equivariant morphism $\phi_f : V_0 \to V^* : v \mapsto \frac{df(v)}{f(v)}$. The weight χ_f determines f up to a scalar. Thus, ϕ_f is, in fact, uniquely determined by χ_f. Moreover, we have $\phi_{f'f''} = \phi_{f'} + \phi_{f''}$ for any two highest weight vectors f', f''. Thus, we can define a map $\mathfrak{a}^* \times V_0 \to V^*$ where $(\chi = \sum_i a_i \lambda_i, v)$ is mapped to $\phi_\chi(v) = \sum_i a_i \phi_{f_i}(v)$, and which has the property $\phi_f(v) = \phi_{\chi_f}(v)$ for all f. For $v \in V_0$ let $\mathfrak{a}^*(v) \subseteq V \oplus V^*$ be the set of points of the form $(v, \phi_\chi(v))$ where χ runs through $\mathfrak{a}^*$. It is an affine subspace canonically isomorphic to $\mathfrak{a}^*$.

4.1 Lemma *For all $b \in B$ we have $\mathfrak{a}^*(bv) = b\mathfrak{a}^*(v)$. Moreover, $G\mathfrak{a}(v)$ is dense in $V \oplus V^*$.*

Proof The first statement follows from the fact that ϕ_χ is B-equivariant. In particular, it does not matter for which point $v \in V_0$ we prove the second assertion. Moreover, the union R of all $\mathfrak{a}(v)$, $v \in V_0$, forms a (trivial) vector bundle over V_0 and equals $B\mathfrak{a}(v)$.

Choose f to be as in Theorem 2.4. Then $\dim V = \operatorname{rk} V + \dim \mathfrak{p}_f^u$. Since $\dim R = \dim V + \operatorname{rk} V$, we obtain that the codimension of R in $V \oplus V^*$ is $\dim V - \operatorname{rk} V = \dim \mathfrak{p}_f^u$.

Consider now the morphism

$$\Psi : V \oplus V^* \to \mathfrak{g}^* : (v, \alpha) \mapsto [\xi \mapsto \alpha(\xi v)].$$

Then we have $\psi_f = -\Phi \circ \phi_f$ (notation as in section 2). Thus Theorem 2.2 implies that $\Phi(R) \subseteq \mathfrak{p}_f$. Moreover, we can find $v \in V_0$ such that $\psi_f(v) = \chi_f'$. Let $\mathfrak{p}_-^u$ be the Lie algebra of the unipotent radical of the opposite parabolic subgroup of P_f. Then, since also the centralizer of χ_f' in $\mathfrak{p}_-^u$ is zero, we have $[\mathfrak{p}_-^u, \chi_f'] = \mathfrak{p}_-^u$. Let $\overline{v} \in V \oplus V^*$ be the point $(v, \phi_f(v))$. Then we obtain that $\mathfrak{p}_-^u \overline{v}$ is transversal to the tangent space of R in $\overline{v}$. By the dimension calculation above we obtain that $G\mathfrak{a}(v) = GR$ is open in $V \oplus V^*$. □

4.2 Corollary *For every $v \in V_0$, restriction to $\mathfrak{a}^*(v)$ defines an injective homomorphism*

$$\overline{c} : (\mathcal{P} \otimes \mathcal{D})^G \to \mathbb{C}[\mathfrak{a}^*] : h \mapsto \overline{c}_h := h|_{\mathfrak{a}^*(v)}.$$

Moreover, $\overline{c}$ does not depend on the choice of v.

Next we do a similar thing for differential operators. Every $D \in (\mathcal{P}\mathcal{D})^G$ acts on each $\mathcal{P}_\lambda$ by multiplication with a scalar $c_D(\lambda)$. Then the map

$$c : (\mathcal{P}\mathcal{D})^G \to \mathbb{C}^\Lambda : D \mapsto (c_D(\lambda))_{\lambda \in \Lambda}$$

is an injective algebra homomorphism. In particular, $(\mathcal{P}\mathcal{D})^G$ is a commutative ring.

4.3 Lemma *Let X be any smooth affine algebraic variety, $f_1, \dots, f_r \in \mathbb{C}[X]$ and D a differential operator on X.*

(a) *There is $b(x; a_1, \dots, a_r) \in \mathbb{C}[X][f_1^{-1}, \dots, f_r^{-1}][a_1, \dots, a_r]$ such that for all $a_i \in \mathbb{Z}$:*

$$D(f_1^{a_1} \dots f_r^{a_r}) = b(x; a_1, \dots, a_r) f_1^{a_1} \dots f_r^{a_r}.$$

Moreover, $\deg_a b \le \operatorname{ord} D$, where $\deg_a$ denotes the degree with respect to the variables a_i.

(b) *For $d = \operatorname{ord} D$ let $\overline{b}$ be the part of b which is homogeneous of degree d in the a_i (might be zero). Let σ_D be the symbol of D, regarded as a function on the cotangent bundle of X. Then*

$$\sigma_D(a_1 \frac{df_1}{f_1} + \dots + a_r \frac{df_r}{f_r}) = \overline{b}(x; a_1, \dots, a_r).$$

Proof We can write $D = g + \sum_i \xi_i E_i$ where g is a function, the ξ_i are vector fields, and the E_i are operators of strictly lower order than D. By induction, the assertion is true for the E_i. Thus we obtain

$$\xi_i E_i(f_1^{a_1} \ldots f_r^{a_r}) = \xi_i(b_i f_1^{a_1} \ldots f_r^{a_r}) = \left(\xi_i(b_i) + b_i \sum_k a_k \frac{\xi_i(f_k)}{f_k}\right) f_1^{a_1} \ldots f_r^{a_r}.$$

This shows (a). Part (b) follows also since $\xi_i(b_i)$ has degree $< d$ in the a_j, and the sum is just the symbol of ξ_i applied to $\sum_k a_k df_k/f_k$. □

4.4 Corollary *The image of c is contained in $\mathbb{C}[\mathfrak{a}^*]$. Moreover, for all $D \in (\mathcal{P}\mathcal{D})^G$ we have $\deg c_D = \operatorname{ord} D$, and the top homogeneous component of c_D equals $\overline{c}_{\sigma_D}$.*

Proof By Theorem 3.2, $\mathcal{P}_\lambda$ contains a unique vector of the form $f_1^{a_1} \ldots f_r^{a_r}$. Then $c_D(\lambda) = D(f_1^{a_1} \ldots f_r^{a_r})/f_1^{a_1} \ldots f_r^{a_r}$. Hence, Lemma 4.3(a) implies that $c_D(\lambda)$ is a polynomial function in λ of degree at most $\operatorname{ord} D$.

The second assertion follows from Lemma 4.3(b), provided the left hand side does not vanish identically. This follows from Lemma 4.1 and the fact that σ_D is G-invariant. □

4.5 Corollary *For all $\lambda \in \Lambda$, $\overline{c}(E_\lambda)$ is the top homogeneous component of $c(D_\lambda)$.*

Since V is a vector space, $\mathcal{P}$ has a natural grading and $\mathcal{P}_\lambda$ consists of homogeneous polynomials. Denote their degree by $|\lambda|$. Clearly, we have $|\lambda + \mu| = |\lambda| + |\mu|$. Analogously, $\mathcal{D}_\lambda$ consists of differential operators of order $|\lambda|$. This shows that E_λ has bidegree $(|\lambda|, |\lambda|)$, and D_λ has order $|\lambda|$.

We define two order relations on Λ: we have $\lambda \le \mu$ if $\mu - \lambda$ is a sum of positive roots. Observe that only weights with the same degree are comparable. To relax this, we define $\lambda \preceq \mu$ if $|\lambda| < |\mu|$ or $\lambda \le \mu$.

4.6 Theorem *For any two weights $\lambda, \mu \in \Lambda$ there are expansions*

$$E_\lambda E_\mu = \sum_{\nu \le \lambda+\mu} a_\nu E_\nu, \qquad D_\lambda D_\mu = \sum_{\nu \preceq \lambda+\mu} a_\nu D_\nu,$$

where $a_{\lambda+\mu}$ is non-zero.

Proof We have $D_\lambda D_\mu \in \mathcal{P}_\lambda \mathcal{D}_\lambda \mathcal{P}_\mu \mathcal{D}_\mu$. Since the commutator lowers both the order of the operators and the degree of the coefficients we obtain

$$D_\lambda D_\mu \in m(E_\lambda E_\mu) + \sum_{|\nu|, |\nu'| < |\lambda+\mu|} \mathcal{P}_\nu \mathcal{D}_{\nu'}.$$

In $\mathcal{P}_\lambda \otimes \mathcal{P}_\mu$ only modules with highest weight $\nu \le \lambda + \mu$ appear. This shows

$$E_\lambda E_\mu \in \sum_{\nu, \nu' \le \lambda+\mu} \mathcal{P}_\nu \otimes \mathcal{D}_{\nu'}.$$

The existence of the claimed expressions follows. Finally, since $f_\lambda f_\mu$ is non-zero and lies in $\mathcal{P}_{\lambda+\mu}$, the highest coefficient map $\kappa : \mathcal{P}_\lambda \otimes \mathcal{P}_\mu \to \mathcal{P}_{\lambda+\mu}$ is non-zero. But it is well known that κ maps no pure tensor $f \otimes g \ne 0$ to zero. One way to see this is: let U be the unipotent radical of G. Then $\mathbb{C}[G]^U$ is the direct sum of all simple G-modules and κ is part of its multiplication law. The claim follows from the fact that $\mathbb{C}[G]^U$ is a domain. □

4.7 Corollary [HoUm] *The algebras $(\mathcal{P} \otimes \mathcal{D})^G$ and $(\mathcal{P}\mathcal{D})^G$ are polynomial rings generated by $E_{\lambda_1}, \ldots, E_{\lambda_r}$ and $D_{\lambda_1}, \ldots, D_{\lambda_r}$, respectively.*

Now we are in the position to determine the image of the homomorphisms c and $\overline{c}$. For this, a change in notation is convenient: let $\varrho \in \mathfrak{t}^*$ be the half-sum of positive roots and consider the affine subspace $\mathfrak{a}^* + \varrho$ of $\mathfrak{t}^*$. Then we get a homomorphism

$$p : (\mathcal{P}\mathcal{D})^G \to \mathbb{C}[\mathfrak{a}^* + \varrho] : D \mapsto p_D,$$

where $p_D(\chi) := c_D(\chi - \varrho)$. This means that D acts on $\mathcal{P}_\lambda$ by the scalar $p_D(\lambda + \varrho)$.

4.8 Theorem *There is a subgroup* W_V *of the Weyl group* $W = W(\mathfrak{g}, \mathfrak{t})$*, stabilizing* $\mathfrak{a}^* + \varrho$*, such that the image of* p *is precisely* $\mathbb{C}[\mathfrak{a}^* + \varrho]^{W_V}$*. The image of* $\overline{c}$ *is* $\mathbb{C}[\mathfrak{a}^*]^{W_V}$*.*

Proof Let $\mathfrak{z}$ be the center of the enveloping algebra of $\mathfrak{g}$. Then the action of $\mathfrak{g}$ on V induces a homomorphism $\mathfrak{z} \to (\mathcal{P}\mathcal{D})^G$. The Harish-Chandra isomorphism establishes an isomorphism $\mathfrak{z} \to \mathbb{C}[\mathfrak{t}^*]^W : \xi \mapsto q_\xi$ such that $\xi \in \mathfrak{z}$ acts on $\mathcal{P}_\lambda$ by multiplication with $q_\xi(-w_0\lambda + \varrho)$. Thus, if we define $\overline{p}_\xi(\lambda) := q_\xi(-w_0\lambda)$ we obtain the following commutative diagram

$$\begin{tikzcd}
\mathfrak{z} \arrow[r, "\overline{p}"] \arrow[d] & \mathbb{C}[\mathfrak{t}^*]^W \arrow[d, "\mathrm{res}"] \\
(\mathcal{P}\mathcal{D})^G \arrow[r, hook] & \mathbb{C}[\mathfrak{a}^* + \varrho]
\end{tikzcd}$$

Let $N \subseteq W$ be the stabilizer of $\mathfrak{a}^* + \varrho$. Then $\operatorname{res} \mathbb{C}[\mathfrak{t}^*]^W \subseteq \mathbb{C}[\mathfrak{a}^* + \varrho]^N$ with equality for the quotient fields. This shows that $\mathbb{C}[\mathfrak{a}^* + \varrho]^N$ is the integral closure of $\operatorname{res} \mathbb{C}[\mathfrak{t}^*]^W$. Since $p((\mathcal{P}\mathcal{D})^G)$ is integrally closed (being a polynomial ring by Theorem 4.8), we get

$$\mathbb{C}[\mathfrak{a}^* + \varrho]^N \subseteq p((\mathcal{P}\mathcal{D})^G) \subseteq \mathbb{C}[\mathfrak{a}^* + \varrho].$$

Hence, by Galois theory, there exists a subgroup $W_V \subseteq N$ such that $p((\mathcal{P}\mathcal{D})^G) = \mathbb{C}[\mathfrak{a}^* + \varrho]^{W_V}$. This shows the first part. The claim on $\overline{c}$ follows from Corollary 4.4. □

4.9 Corollary *The little Weyl group* W_V *acts as reflection group on* $\mathfrak{a}^*$*. Moreover, the degrees of the generating invariants of* W_V *equal the degrees of the* f_i *from Theorem* 3.2.

Proof This follows from the theorem of Shephard-Todd-Chevalley since $\mathbb{C}[\mathfrak{a}^*]^{W_V} \cong \mathbb{C}[\mathfrak{a}^* + \varrho]^{W_V}$ is a polynomial ring. □

Remark A much more general version of the theorem, valid for all G-varieties, was first proved in [Knop2]. For smooth affine G-varieties it states that the center of the ring of invariant differential operators is a polynomial ring which is canonically isomorphic to the ring of invariants of a finite reflection group.

The polynomials $\overline{p}_\lambda := \overline{c}(E_\lambda)$ and $p_\lambda := p(D_\lambda)$ form a basis of $\mathbb{C}[\mathfrak{a}^*]^{W_V}$ and $\mathbb{C}[\mathfrak{a}^* + \varrho]^{W_V}$, respectively. For the latter, we give a very simple internal characterization. This was suggested to me by Sahi who had proved a special case beforehand [Sahi]. Parts of it can be found in [HoUm].

4.10 Theorem *Let* $\lambda \in \Lambda$*. Then* p_λ *is, up to a scalar, the unique non-zero polynomial function on* $\mathfrak{a}^* + \varrho$ *having the following properties:*

(1) p_λ *is* W_V*-invariant;*

(2) $\deg p_\lambda \le |\lambda|$*;*

(3) $p_\lambda(\mu + \varrho) = 0$ *for all* $\mu \in \Lambda$ *with* $|\mu| \le |\lambda|$ *and* $\mu \ne \lambda$*.*

Moreover, p_λ *has the additional property* $p_\lambda(\lambda + \varrho) \ne 0$*.*

Proof First we show that p_λ has these properties. (1) holds by Theorem 4.8 and (2) by Corollary 4.4. For (3) observe that $D_\lambda \in \mathcal{P}_\lambda \cdot \mathcal{D}_\lambda$. The action of $\mathcal{PD}$ on $\mathcal{P}$ gives a map $s : \mathcal{D}_\lambda \otimes \mathcal{P}_\mu \to \mathcal{P}$. Since $\mathcal{D}$ consists of the constant coefficient differential operators, the image of s is contained in the space $\mathcal{P}^d$ of homogeneous polynomials of degree $d = |\mu| - |\lambda|$. Hence $s = 0$ for $|\mu| < |\lambda|$. If $|\mu| = |\lambda|$ then s is a G-invariant linear form, hence zero for $\mu \neq \lambda$. This shows (3).

Next we show $p_\lambda(\lambda + \varrho) \neq 0$ or equivalently that D_λ acts non-trivially on $\mathcal{P}_\lambda$. Let f_i be a basis of $\mathcal{P}_\lambda$ consisting of weight vectors. Then one of them, say f_1, is a highest weight vector. Let $d_i \in \mathcal{D}_\lambda$ be a dual basis. Then D_λ can we written as $\sum_i f_i d_i$. The operator d_i has constant coefficients. Hence $d_i(f_1)$ is a polynomial of degree zero, hence constant. On the other hand, its weight is $\chi_{d_i} + \chi_{f_1}$. Thus, $d_i(f_1) = 0$ unless $i = 1$. We have already seen that D_λ kills all $\mathcal{P}_\mu$ with $|\mu| = |\lambda|$ and $\mu \neq \lambda$. Thus d_1 cannot act trivially on $\mathcal{P}_\lambda$, hence $D_\lambda(f_1) = f_1 d_1(f_1) \neq 0$, which shows the claim.

For uniqueness set $d = |\lambda|$, let $\mathcal{F}_d \subseteq \mathbb{C}[\mathfrak{a}^* + \varrho]^{W_V}$ be the subspace of polynomials of degree at most d and $\Lambda_d := \{\mu \in \Lambda \mid |\mu| \leq d\}$. Then we get an evaluation map

$$\varepsilon : \mathcal{F}_d \to \mathbb{C}^{\Lambda_d} : p \mapsto (p(\mu + \varrho))_{\mu \in \Lambda_d}$$

By (1) and (2), we have $p_\mu \in \mathcal{F}_d$ for all $\mu \in \Lambda_d$. By (3) and $p_\lambda(\lambda + \varrho) \neq 0$, their images $\varepsilon(p_\mu)$ form a basis of $\mathbb{C}^{\Lambda_d}$. This shows that ε is surjective. By Corollary 4.7, the algebra $\mathbb{C}[\mathfrak{a}^* + \varrho]^{W_V} \cong (\mathcal{PD})^G$ is a polynomial ring generated by $p_i := p_{D_{\lambda_i}}$. The degree of p_i is $|\lambda_i|$ (Corollary 4.4). Furthermore, Corollary 4.7 and Theorem 3.2(2) imply that the highest degree parts of $p_1, \ldots, p_r$ are algebraically independent. This shows that $\mathcal{F}_d$ has a basis consisting of all $p_1^{a_1} \ldots p_r^{a_r}$ with $a_i \in \mathbb{N}$ and $\sum_i a_i |\lambda_i| \leq d$. We conclude that $\dim \mathcal{F}_d = \#\Lambda_d = \dim \mathbb{C}^{\Lambda_d}$. Hence ε is also injective which means that p_λ is unique. ☐

In principle there is also a characterization for the image $\overline{p}_\lambda$ of a spherical function E_λ (other than via p_λ):

4.11 Theorem *For every $h \in \mathbb{C}[\mathfrak{a}^* + \varrho]^{W_V}$ there is a unique differential operator $\mathbb{D}_h$ on $\mathbb{C}[\mathfrak{a}^*]^{W_V}$ such that $\mathbb{D}_h(\overline{p}_\lambda) = h(\lambda + \varrho)\overline{p}_\lambda$.*

Proof This is almost a tautology. The polynomial h corresponds to a differential operator $D_h \in (\mathcal{PD})^G$. We let D_h act on $\mathcal{P} \otimes \mathcal{D}$ by applying it to the left factor. Then D_h is a differential operator on $V \oplus V^*$ which acts as scalar $h(\lambda + \varrho)$ on each space $\mathcal{P}_\lambda \otimes \mathcal{D}_\mu$. In particular, $D_h(E_\lambda) = h(\lambda + \varrho) E_\lambda$. Thus D_h restricts to a differential operator $(\mathcal{P} \otimes \mathcal{D})^G = \mathbb{C}[\mathfrak{a}^*]^{W_V}$ with the required property. ☐

Clearly, the map $h \to \mathbb{D}_h$ is an injective algebra homomorphism. In particular, the $\mathbb{D}_h$'s commute pairwise. If one knows the operators $\mathbb{D}_{p_{\lambda_1}}, \ldots, \mathbb{D}_{p_{\lambda_r}}$ explicitly, then one can characterize the $\overline{p}_\lambda$ as their common eigenvectors. Sekiguchi has computed them in special cases ($\mathbb{C}^m \otimes \mathbb{C}^n$, $\Lambda^2\mathbb{C}^n$, $S^2\mathbb{C}^n$, and E_6). In these cases, it can be shown that these differential operators lift to difference operators having the p_λ as common eigenvectors [KnSa].

5 The classification of multiplicity free representations

Multiplicity free representations (G, V) are now completely classified. Kac [Kac] determined all cases where V is irreducible. Brion [Brion] did the case where (G, G) is (almost) simple. Finally, Leahy [Leahy] and Benson-Ratcliff [BenRat] did the rest, independently.

To make the problem manageable one needs some simple concepts. If (G, V) is multiplicity free and $\overline{G} \to G$ a surjective homomorphism, then we obtain another multiplicity free representation $(\overline{G}, V)$. Therefore, we make the following

Definition Two representations $\varrho_1 : G_1 \to GL(V_1)$ and $\varrho_2 : G_2 \to GL(V_2)$ are called *geometrically equivalent* if there is an isomorphism $\phi : V_1 \to V_2$, inducing $GL(\phi) : GL(V_1) \to GL(V_2)$, such that $GL(\phi)(\varrho_1(G_1)) = \varrho_2(G_2)$.

For example, the representation of $SL_2(\mathbb{C})$ on $S^2(\mathbb{C})$ is geometrically equivalent to the representation of $SO_3(\mathbb{C})$ on $\mathbb{C}^3$. Observe also, that every representation is geometrically equivalent to its dual representation.

Another way to produce new multiplicity free representations from old ones is: let (G_1, V_1) and (G_2, V_2) be multiplicity free. Then also $(G_1 \times G_2, V_1 \oplus V_2)$ is multiplicity free. Hence:

Definition A representation (G, V) is *decomposable* if it is geometrically equivalent to a representation of the form $(G_1 \times G_2, V_1 \oplus V_2)$ with non-zero V_1 and V_2. It is called *indecomposable* if it is not decomposable.

There is still a problem: for $n \geq 1$ the representation of $SL_n(\mathbb{C})$ on $\mathbb{C}^n$ is multiplicity free. From this we can produce infinitely many multiplicity-representations as follows: choose $k \geq 1$ and take $V = (\mathbb{C}^n)^{\oplus k}$ and $G = \mathbb{C}^* \times SL_n(\mathbb{C})^k$. Here each SL_n-factor acts on the corresponding $\mathbb{C}^n$-summand. We let $\mathbb{C}^*$ act on the i-th summand by multiplication with t^{a_i}, where $a_1, \ldots, a_k$ are arbitrary integers. If none of the a_i are zero then (G, V) is indecomposable. Thus there are indecomposable multiplicity free representations with arbitrarily many irreducible summands. Thus we define:

Definition A representation $\varrho : G \to GL(V)$ is called *saturated* if the dimension of the center of $\varrho(G)$ equals the number of irreducible summands of V.

Clearly, every representation can be made saturated by adding a sufficiently big torus. Thus, the following theorem reduces the classification of multiplicity free representations to the saturated ones. A proof can be found in [Leahy].

5.1 Theorem *Let* (G, V) *be a multiplicity free representation and let* $G_0 \subseteq G$ *be a connected subgroup containing* (G, G). *Let* $\mathfrak{z}$ *be the center of* $\mathfrak{g}$, *and* $\mathfrak{z}_0$ *the center of* $\mathfrak{g}_0$. *Observe that both* $\mathfrak{a}^*$ *and* $\mathfrak{z}^*$ *can be considered as subspaces of* $\mathfrak{t}^*$, *where* $\mathfrak{t}$ *is a Cartan subalgebra of* $\mathfrak{g}$, *and that there is a restriction map* $\mathfrak{z}^* \twoheadrightarrow \mathfrak{z}_0^*$. *Then* (G_0, V) *is multiplicity free if and only if* $\mathfrak{a}^* \cap \mathfrak{z}^* \to \mathfrak{z}_0^*$ *is injective. In other words,* G_0 *is multiplicity free if and only if its center separates the weights of* $\mathfrak{a}^* \cap \mathfrak{z}^*$.

The set $\mathfrak{a}^* \cap \mathfrak{z}^*$ is given below in the tables.

5.2 Theorem *Below is a complete (but somewhat redundant) list of indecomposable, saturated, multiplicity free representations up to geometric equivalence.*

Explanation: The first item of each entry is the group G and the space V on which it acts. In most cases it should be pretty clear how G acts on V. If not, the highest weights can be obtained as the basic weights of degree one. Next we note the rank, i.e., the dimension of $\mathfrak{a}^*$.

The basic weights are the indecomposable generators of Λ. They are taken from [Leahy] with some modifications. We use the following conventions: ω, ω', ω'' refer to weights of the non-abelian factors, while ε, ε' denote characters of $\mathbb{C}^*$-factors. For $GL_n(\mathbb{C})$ we denote the highest weight of $\Lambda^k(\mathbb{C}^n)$ by ω_k. For all other groups, ω_k denotes a fundamental weight in Bourbaki-numbering.

The entry "degree functions" allows the calculation of $|\lambda|$ for $\lambda \in \Lambda$. In some cases it was convenient to define the degree function on a larger set. Then it is not quite canonical.

The significance of $\mathfrak{a}^* \cap \mathfrak{z}^*$ is explained in Theorem 5.1.

Finally, we list the simple reflections of W_V. We indicate after the table how to compute them. Again, primes s, s', etc., are used to refer to the various non-abelian factors. The reflections in the Weyl group of $GL_n(\mathbb{C})$ are denoted by s_{ij}. For the other groups (except $SL_2(\mathbb{C})$ where we use s) we use s_α where α is a positive root. For the classical groups α is given in the usual ε_i-basis (see Bourbaki).

Irreducible representations

$GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ on $\mathbb{C}^m \otimes \mathbb{C}^n$ with $1 \le m \le n$

Rank: m

Basic weights: $\omega_i + \omega_i'$, $i = 1, \ldots, m$.

Degree function: $|\omega_i + \omega_i'| = i$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: $\begin{cases} \emptyset, & \text{if } m < n \\ \omega_m + \omega_m', & \text{if } m = n \end{cases}$

Simple reflections of W_V: $s_{i\,i+1}s'_{i\,i+1}$, $i = 1, \ldots, m-1$.

$GL_n(\mathbb{C})$ on $S^2(\mathbb{C}^n)$ with $1 \le n$

Rank: n

Basic weights: $2\omega_i$, $i = 1, \ldots, n$.

Degree function: $|2\omega_i| = i$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: $2\omega_n$

Simple reflections of W_V: $s_{i\,i+1}$, $i = 1, \ldots, n-1$.

$GL_n(\mathbb{C})$ on $\Lambda^2(\mathbb{C}^n)$ with $2 \le n$

Rank: $\lfloor \frac{n}{2} \rfloor$

Basic weights: ω_{2i}, $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$.

Degree function: $|\omega_{2i}| = i$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: $\begin{cases} \emptyset, & \text{if } n \text{ odd} \\ \omega_n, & \text{if } n \text{ even} \end{cases}$

Simple reflections of W_V: $s_{2i-1\,2i+1}s_{2i\,2i+2}$, $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1$.

$Sp_{2n}(\mathbb{C}) \times \mathbb{C}^*$ on $\mathbb{C}^{2n}$ with $1 \le n$

Rank: 1

Basic weights: $\omega_1 + \varepsilon$.

Degree function: $|\omega_1 + \varepsilon| = 1$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: $\emptyset$

Simple reflections of W_V: $\emptyset$.

$Sp_{2n}(\mathbb{C}) \times GL_2(\mathbb{C})$ on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$ with $2 \leq n$

Rank: 3

Basic weights: $\omega_1 + \omega_1', \omega_2 + \omega_2', \omega_2'$.

Degree function: $|\omega_i| = 0$, $|\omega_i'| = i$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: ω_2'

Simple reflections of W_V: $s_{\varepsilon_1 - \varepsilon_2} s_{12}'$, $s_{\varepsilon_1 + \varepsilon_2}$.

$Sp_{2n}(\mathbb{C}) \times GL_3(\mathbb{C})$ on $\mathbb{C}^{2n} \otimes \mathbb{C}^3$ with $3 \leq n$

Rank: 6

Basic weights: $\omega_1 + \omega_1', \omega_2 + \omega_2', \omega_3 + \omega_3', \omega_2', \omega_1 + \omega_3', \omega_2 + \omega_1' + \omega_3'$.

Degree function: $|\omega_i| = 0$, $|\omega_i'| = i$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: $2\omega_3'$

Simple reflections of W_V: $s_{\varepsilon_1 - \varepsilon_2}, s_{\varepsilon_2 - \varepsilon_3}, s_{\varepsilon_2 + \varepsilon_3}, s'_{\varepsilon_1 - \varepsilon_2}, s'_{\varepsilon_2 - \varepsilon_3}$.

$Sp_4(\mathbb{C}) \times GL_3(\mathbb{C})$ on $\mathbb{C}^4 \otimes \mathbb{C}^3$

Rank: 5

Basic weights: $\omega_1 + \omega_1', \omega_2 + \omega_2', \omega_2', \omega_1 + \omega_3', \omega_2 + \omega_1' + \omega_3'$.

Degree function: $|\omega_i| = 0$, $|\omega_i'| = i$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: $2\omega_3'$

Simple reflections of W_V: $s_{\varepsilon_1 - \varepsilon_2}, s_{2\varepsilon_2}, s'_{\varepsilon_1 - \varepsilon_2}, s'_{\varepsilon_2 - \varepsilon_3}$.

$Sp_4(\mathbb{C}) \times GL_n(\mathbb{C})$ on $\mathbb{C}^4 \otimes \mathbb{C}^n$ with $4 \leq n$

Rank: 6

Basic weights: $\omega_1 + \omega_1', \omega_2 + \omega_2', \omega_2', \omega_1 + \omega_3', \omega_2 + \omega_1' + \omega_3', \omega_4'$.

Degree function: $|\omega_i| = 0$, $|\omega_i'| = i$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: $\begin{cases} \emptyset, & \text{if } 4 < n \\ \omega_4', & \text{if } 4 = n \end{cases}$

Simple reflections of W_V: $s_{\varepsilon_1 - \varepsilon_2}, s_{2\varepsilon_2}, s'_{\varepsilon_1 - \varepsilon_2}, s'_{\varepsilon_2 - \varepsilon_3}, s'_{\varepsilon_3 - \varepsilon_4}$.

$SO_n(\mathbb{C}) \times \mathbb{C}^*$ on $\mathbb{C}^n$ with $3 \leq n$.

Rank: 2

Basic weights: $\omega_1 + \varepsilon, 2\varepsilon$.

Degree function: $|\omega_1| = 0$, $|\varepsilon| = 1$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: 2ε

Simple reflections of W_V: $\begin{cases} s_{\varepsilon_1}, & \text{if } n \text{ odd} \\ s_{\varepsilon_1} s_{\varepsilon_n}, & \text{if } n \text{ even} \end{cases}$.

$Spin_{10}(\mathbb{C}) \times \mathbb{C}^*$ on $\mathbb{C}^{16}$

Rank: 2

Basic weights: $\omega_5 + \varepsilon, \omega_1 + 2\varepsilon$.

Degree function: $|\omega_i| = 0$, $|\varepsilon| = 1$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: $\emptyset$

Simple reflections of W_V: $s_{\varepsilon_2 + \varepsilon_5} s_{\varepsilon_3 + \varepsilon_4}$.

$Spin_7(\mathbb{C}) \times \mathbb{C}^*$ on $\mathbb{C}^8$

Rank: 2

Basic weights: $\omega_3 + \varepsilon, 2\varepsilon$.

Degree function: $|\omega_3| = 0$, $|\varepsilon| = 1$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: 2ε

Simple reflections of W_V: $s_{\varepsilon_1+\varepsilon_3}s_{\varepsilon_2}$

$Spin_9(\mathbb{C}) \times \mathbb{C}^*$ on $\mathbb{C}^{16}$

Rank: 3

Basic weights: $\omega_4 + \varepsilon, \omega_1 + 2\varepsilon, 2\varepsilon$.

Degree function: $|\omega_i| = 0$, $|\varepsilon| = 1$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: 2ε

Simple reflections of W_V: $s_{\varepsilon_1}, s_{\varepsilon_2+\varepsilon_4}s_{\varepsilon_3}$.

$\mathrm{G}_2 \times \mathbb{C}^*$ on $\mathbb{C}^7$

Rank: 2

Basic weights: $\omega_1 + \varepsilon, 2\varepsilon$.

Degree function: $|\omega_i| = 0$, $|\varepsilon| = 1$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: 2ε

Simple reflections of W_V: s_{ω_1}.

$\mathrm{E}_6 \times \mathbb{C}^*$ on $\mathbb{C}^{27}$

Rank: 3

Basic weights: $\omega_1 + \varepsilon, \omega_6 + 2\varepsilon, 3\varepsilon$.

Degree function: $|\omega_i| = 0$, $|\varepsilon| = 1$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: 3ε

Simple reflections of W_V : $\begin{cases} s_{\beta_1}s_{\beta_2}, & \text{where } \beta_1{=}\alpha_3{+}\alpha_4{+}\alpha_5{+}\alpha_6, \quad \beta_2{=}\alpha_2{+}\alpha_4{+}\alpha_5{+}\alpha_6, \\ s_{\gamma_1}s_{\gamma_2}, & \text{where } \gamma_1{=}\alpha_1{+}\alpha_2{+}\alpha_3{+}\alpha_4, \quad \gamma_2{=}\alpha_1{+}\alpha_3{+}\alpha_4{+}\alpha_5. \end{cases}$

Reducible representations

$GL_n(\mathbb{C}) \times \mathbb{C}^*$ on $\Lambda^2(\mathbb{C}^n) \oplus \mathbb{C}^n$ with $4 \leq n$

Rank: n

Basic weights: $\begin{cases} \omega_{2i-1}{+}\varepsilon, & i = 1, \dots, \lceil \frac{n}{2} \rceil \\ \omega_{2i}, & i = 1, \dots, \lfloor \frac{n}{2} \rfloor \end{cases}$

Degree function: $|\omega_i| = \frac{i}{2}$, $|\varepsilon| = \frac{1}{2}$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: $\begin{cases} \omega_n + \varepsilon, & \text{if } n \text{ odd} \\ \omega_n, & \text{if } n \text{ even} \end{cases}$

Simple reflections of W_V: $s_{i\,i+2}$, $i = 1, \dots, n-2$.

$GL_n(\mathbb{C}) \times \mathbb{C}^*$ on $\Lambda^2(\mathbb{C}^n) \oplus (\mathbb{C}^n)^*$ with $4 \leq n$

Rank: n

Basic weights: $\begin{cases} \omega_{2i-1} + \varepsilon, & i = 1, \dots, \lceil \frac{n}{2} \rceil - 1 \\ \omega_{n-1} - \omega_n + \varepsilon, & \\ \omega_{2i}, & i = 1, \dots, \lfloor \frac{n}{2} \rfloor \end{cases}$

Degree function: $|\omega_i| = \frac{i}{2}$, $|\varepsilon| = \frac{3}{2}$.

Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: $\begin{cases} \omega_n - \varepsilon, & \text{if } n \text{ odd} \\ \omega_n, & \text{if } n \text{ even} \end{cases}$

Simple reflections of W_V : $\begin{cases} s_{i\,i+2}, & i=1,\dots,2\lfloor\frac{n}{2}\rfloor-2 \\ s_{n-1\,n}, & \text{if } n \text{ odd.} \end{cases}$

$GL_m(\mathbb{C})\times GL_n(\mathbb{C})$ on $(\mathbb{C}^m\otimes\mathbb{C}^n)\oplus\mathbb{C}^n$ with $2\le m,n$

Rank: $\min(2m+1,2n)$

Basic weights: $\begin{cases} \omega_{i-1}+\omega'_i, & i=1,\dots,\min(m+1,n) \quad (\text{with } \omega_0:=0) \\ \omega_i+\omega'_i, & i=1,\dots,\min(m,n) \end{cases}$

Degree function: $|\omega_i|=0$, $|\omega'_i|=i$.

Basis of $\mathfrak{a}^*\cap\mathfrak{z}^*$: $\begin{cases} \omega_m, & \text{if } m+1<n \\ \omega_m,\omega'_n, & \text{if } m\le n\le m+1 \\ \omega'_n, & \text{if } n<m \end{cases}$

Simple reflections of W_V : $\begin{cases} s_{i\,i+1}, & i=1,\dots,\min(m,n)-1, \\ s'_{i\,i+1}, & i=1,\dots,\min(m+1,n)-1. \end{cases}$

$GL_m(\mathbb{C})\times GL_n(\mathbb{C})$ on $(\mathbb{C}^m\otimes\mathbb{C}^n)\oplus(\mathbb{C}^n)^*$ with $2\le m,n$

Rank: $\min(2m+1,2n)$

Basic weights: $\begin{cases} \omega_i+\omega'_{i-1}, & i=1,\dots,\min(m,n-1) \quad (\text{with } \omega'_0:=0) \\ \omega'_{n-1}-\omega'_n, & \\ \omega_i+\omega'_i, & i=1,\dots,\min(m,n) \end{cases}$

Degree function: $|\omega_i|=2i$, $|\omega'_i|=-i$.

Basis of $\mathfrak{a}^*\cap\mathfrak{z}^*$: $\begin{cases} \omega_m, & \text{if } m+1<n \\ \omega_m,\omega'_n, & \text{if } m\le n\le m+1 \\ \omega'_n, & \text{if } n<m \end{cases}$

Simple reflections of W_V : $\begin{cases} s_{i\,i+1}, & i=1,\dots,\min(m,n)-1, \\ s'_{i\,i+1}, & i=1,\dots,\min(m,n)-1, \\ s'_{m\,n}, & \text{if } m<n. \end{cases}$

$Sp_{2n}\times\mathbb{C}^*\times\mathbb{C}^*$ on $\mathbb{C}^{2n}\oplus\mathbb{C}^{2n}$ with $2\le n$

Rank: 4

Basic weights: $\omega_1+\varepsilon,\omega_1+\varepsilon',\omega_2+\varepsilon+\varepsilon',\varepsilon+\varepsilon'$.

Degree function: $|\omega_i|=0$, $|\varepsilon|=1$, $|\varepsilon'|=1$.

Basis of $\mathfrak{a}^*\cap\mathfrak{z}^*$: ε, ε'

Simple reflections of W_V: $s_{\varepsilon_1-\varepsilon_2}$, $s_{\varepsilon_1+\varepsilon_2}$.

$(Sp_{2n}(\mathbb{C})\times\mathbb{C}^*)\times GL_2(\mathbb{C})$ on $(\mathbb{C}^{2n}\otimes\mathbb{C}^2)\oplus\mathbb{C}^2$ with $2\le n$

Rank: 5

Basic weights: $\omega'_1,\omega_1+\varepsilon+\omega'_1,\omega_1+\varepsilon+\omega'_2,\omega_2+2\varepsilon+\omega'_2,2\varepsilon+\omega'_2$.

Degree function: $|\omega_1|=0$, $|\varepsilon|=0$, $|\omega'_i|=i$.

Basis of $\mathfrak{a}^*\cap\mathfrak{z}^*$: ε,ω'_2.

Simple reflections of W_V: $s_{\varepsilon_1-\varepsilon_2}$, $s_{\varepsilon_1+\varepsilon_2}$, s'_{12}.

$GL_m(\mathbb{C})\times SL_2(\mathbb{C})\times GL_n(\mathbb{C})$ on $(\mathbb{C}^m\otimes\mathbb{C}^2)\oplus(\mathbb{C}^2\otimes\mathbb{C}^n)$ with $2\le m\le n$

Rank: 5

Basic weights: $\omega_1+\omega',\omega'+\omega''_1,\omega_1+\omega''_1,\omega_2,\omega''_2$.

Degree function: $|\omega_i|=i$, $|\omega'|=0$, $|\omega''_i|=i$.

Basis of $\mathfrak{a}^*\cap\mathfrak{z}^*$: $\begin{cases} \emptyset, & \text{if } 2<m\le n \\ \omega_2, & \text{if } 2=m<n \\ \omega_2,\omega''_2, & \text{if } 2=m=n \end{cases}$

Simple reflections of W_V: s_{12}, s', s''_{12}.

$(Sp_{2m}(\mathbb{C}) \times \mathbb{C}^*) \times SL_2(\mathbb{C}) \times GL_n(\mathbb{C})$ on $(\mathbb{C}^{2m} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^n)$ with $2 \leq m, n$
Rank: 6
Basic weights: $\omega_1 + \varepsilon + \omega', \omega' + \omega''_1, \omega_1 + \varepsilon + \omega''_1, \omega_2 + 2\varepsilon, \omega''_2, 2\varepsilon$.
Degree function: $|\omega_i| = 0$, $|\varepsilon| = 1$, $|\omega'| = 0$, $|\omega''_i| = i$.
Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: $\begin{cases} 2\varepsilon, & \text{if } 2 < n \\ 2\varepsilon, \omega''_2, & \text{if } 2 = n \end{cases}$
Simple reflections of W_V: $s_{\varepsilon_1-\varepsilon_2}$, $s_{\varepsilon_1+\varepsilon_2}$, s', s''_{12}.

$(Sp_{2m}(\mathbb{C}) \times \mathbb{C}^*) \times SL_2(\mathbb{C}) \times (Sp_{2n}(\mathbb{C}) \times \mathbb{C}^*)$ on $(\mathbb{C}^{2m} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2n})$ with $2 \leq m, n$
Rank: 7
Basic weights: $\omega_1 + \varepsilon + \omega', \omega' + \omega''_1 + \varepsilon', \omega_1 + \varepsilon + \omega''_1 + \varepsilon', \omega_2 + 2\varepsilon, \omega''_2 + 2\varepsilon', 2\varepsilon, 2\varepsilon'$.
Degree function: $|\omega_i| = 0$, $|\varepsilon| = 1$, $|\omega'| = 0$, $|\omega''_i| = 0$, $|\varepsilon'| = 1$.
Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: 2ε, $2\varepsilon'$.
Simple reflections of W_V: $s_{\varepsilon_1-\varepsilon_2}$, $s_{\varepsilon_1+\varepsilon_2}$, s', $s''_{\varepsilon_1-\varepsilon_2}$, $s''_{\varepsilon_1+\varepsilon_2}$.

$Spin_8(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{C}^8_+ \oplus \mathbb{C}^8_-$
Rank: 5
Basic weights: $\omega_3 + \varepsilon, \omega_4 + \varepsilon', \omega_1 + \varepsilon + \varepsilon', 2\varepsilon, 2\varepsilon'$.
Degree function: $|\omega_i| = 0$, $|\varepsilon| = 1$, $|\varepsilon'| = 1$.
Basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$: 2ε, $2\varepsilon'$
Simple reflections of W_V: $s_{\varepsilon_1-\varepsilon_4}$, $s_{\varepsilon_1+\varepsilon_4}$, $s_{\varepsilon_2+\varepsilon_3}$.

Now we indicate how to compute the Weyl groups. They have to meet two requirements:

(a) W_V stabilizes $\mathfrak{a}^* + \varrho$.

(b) The degrees of the basic invariants of W_V equal the degrees of the basic weights.

In particular, from (b) one can calculate the order of W_V. In most cases this order equals the order of the normalizer of $\mathfrak{a}^* + \varrho$ in W and we are done. In the case $Sp_{2n} \otimes GL_3$ one obtains that W_V is of index 2 in the normalizer which is a reflection group of type $\mathrm{C}_3 \times \mathrm{A}_2$. Luckily, there is only one such subgroup generated by reflections, namely $\mathrm{D}_3 \times \mathrm{A}_2$. The only cases which remain ambiguous are all the reducible representations involving a symplectic group. The problem is that C_2 has two reflection subgroups of index two: reflections about the short and reflections about the long roots. With the following lemma we will be able to show that in fact it is always the short roots.

5.3 Lemma *Let (L, N) be as in the proof of Theorem 3.3. Then N is a multiplicity free representation of L and we have $W_N \subseteq W_V$.*

Proof The map $P^u_f \times \Sigma \to V$ is an open embedding. Thus we obtain a restriction map from G-invariant differential operators on V to L-invariant differential operators on Σ. Moreover, Σ is an open subset of N. Since $\mathbb{C}[N]$ is a submodule of $\mathbb{C}[\Sigma]$ and the latter has no multiplicities we see that every L-invariant differential operator on Σ extends to N. It is easy to see that this restriction map is compatible with the Harish-Chandra isomorphism. Thus we obtain $\mathbb{C}[\mathfrak{a}^* + \varrho]^{W_V} \subseteq \mathbb{C}[\mathfrak{a}^* + \varrho]^{W_N}$, i.e., $W_V \supseteq W_N$. □

Now we can settle all other cases: Taking $V = Sp_{2n} \otimes GL_3$ (an unambiguous case, see above) and χ the highest weight we obtain essentially $L = Sp_{2n-2} \times GL_2$ acting on $N = (\mathbb{C}^{2n-2} \otimes \mathbb{C}^2) \oplus \mathbb{C}^2$. This proves the claim for (L, N). Applying this procedure a second time with $\chi = -\varepsilon_1 + \varepsilon_2'$ one gets essentially the representation of $L' = Sp_{2n-2} \times \mathbb{C}^* \times \mathbb{C}^*$ on $N' = \mathbb{C}^{2n-2} \oplus \mathbb{C}^{2n-2}$. Thus also that representation is settled. Finally, (L', N') is also obtained from all other ambiguous cases by this procedure which settles them all.

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