

Classification of multiplicity free symplectic representations

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Abstract. Let G be a connected reductive group acting on a finite dimensional vector space V . Assume that V is equipped with a G -invariant symplectic form. Then the ring $\mathcal{O}(V)$ of polynomial functions becomes a Poisson algebra. The ring $\mathcal{O}(V)^G$ of invariants is a sub-Poisson algebra. We call V multiplicity free if $\mathcal{O}(V)^G$ is Poisson commutative, i.e., if $\{f, g\} = 0$ for all invariants f and g . Alternatively, G also acts on the Weyl algebra $\mathcal{W}(V)$ and V is multiplicity free if and only if the subalgebra $\mathcal{W}(V)^G$ of invariants is commutative. In this paper we classify all multiplicity free symplectic representations.

1. Introduction

Let V be a finite dimensional complex vector space equipped with a symplectic form ω . Then we can form the *Weyl algebra* $\mathcal{W}(V)$ which, by definition, is generated by V with relations

$$(1.1) \quad v \cdot w - w \cdot v = \omega(v, w).$$

Observe that the symplectic group $Sp(V)$ acts on $\mathcal{W}(V)$ by algebra automorphisms.

Assume a connected reductive group G is acting linearly on V preserving the symplectic structure ω . Then V is called a *symplectic representation* of G . One can think of a symplectic representation as a homomorphism $\rho : G \rightarrow Sp(V)$. Hence G will also act on the Weyl algebra $\mathcal{W}(V)$.

Definition: The symplectic representation $G \rightarrow Sp(V)$ is called *multiplicity free* (or a “MFSR”) if $\mathcal{W}(V)^G$ is commutative.

Another way to phrase the definition is as follows. Recall (see, e.g., [Ho]) that there is a Lie algebra homomorphism $\mathfrak{sp}(V) \hookrightarrow \mathcal{W}(V)$ whose adjoint action integrates to the $Sp(V)$ -

action. Thus we get a Lie algebra homomorphism

$$(1.2) \quad \tilde{\rho} : \mathfrak{g} \rightarrow \mathcal{W}(V)$$

such that $\mathcal{W}(V)^G$ is simply the commutant of $\tilde{\rho}(\mathfrak{g})$ in $\mathcal{W}(V)$.

There are two instances of multiplicity free symplectic representations which have been studied extensively. The first one are Howe's reductive dual pairs [Ho]. More precisely, let

$$(1.3) \quad G = G_1 \times G_2 = Sp_{2m}(\mathbb{C}) \times SO_n(\mathbb{C}) \text{ acting on } V = \mathbb{C}^{2m} \otimes \mathbb{C}^n \quad \text{or}$$

$$(1.4) \quad G = G_1 \times G_2 = GL_m(\mathbb{C}) \times GL_n(\mathbb{C}) \text{ acting on } V = (\mathbb{C}^m \otimes \mathbb{C}^n) \oplus (\mathbb{C}^m \otimes \mathbb{C}^n)^*$$

Then Howe showed that the algebras generated by \mathfrak{g}_1 and \mathfrak{g}_2 are mutual commutants¹ inside $\mathcal{W}(V)$. In particular, the commutant of \mathfrak{g} is commutative.

The other class of multiplicity free symplectic representations studied previously generalizes the case (1.4) above. More precisely, let $G \rightarrow GL(U)$ be any finite dimensional representation. Then $V = U \oplus U^*$ carries the G -invariant symplectic structure

$$(1.5) \quad \omega(u_1 + u_1^*, u_2 + u_2^*) = \langle u_1^*, u_2 \rangle - \langle u_2^*, u_1 \rangle.$$

In this situation one can identify $\mathcal{W}(V)$ with the algebra $\mathcal{PD}(U)$ of polynomial coefficient linear differential operators on U . From that fact one can deduce that (G, V) is multiplicity free if and only if the algebra $\mathbb{C}[U]$ of polynomial functions on U is multiplicity free, i.e., does not contain any simple G -module more than once. Modules U of this type are also called *multiplicity free spaces*. They have been thoroughly investigated, e.g., in [HU]. In particular, they have been classified by Kac [Ka], Benson-Ratcliff [BR], and Leahy [Le]. In the present paper we extend this work by classifying all multiplicity free symplectic representations

We continue by stating some fundamental properties of multiplicity free symplectic representations, proved in [Kn]. Let $\mathcal{Z}(\mathfrak{g})$ be the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Then the homomorphism (1.2) induces algebra homomorphisms

$$(1.6) \quad \mathcal{U}(\tilde{\rho}) : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{W}(V) \quad \text{and} \quad \mathcal{U}(\tilde{\rho})^G : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathcal{W}(V)^G.$$

Clearly, if $\mathcal{U}(\tilde{\rho})^G$ is surjective then V is multiplicity free. In general, one can show that V is a MFSR if and only if $\mathcal{U}(\tilde{\rho})^G$ is "almost" surjective, i.e., $\mathcal{W}(V)^G$ is a finitely generated $\mathcal{Z}(\mathfrak{g})$ -module.

¹ In situation (1.3) this is not quite true. But still the commutant of \mathfrak{g}_1 is the algebra generated by \mathfrak{g}_2 which suffices for our purposes.

The Weyl algebra is filtered by putting V in degree 1. The associated graded version of $\mathcal{U}(\tilde{\rho})$ is the *moment map*¹

$$(1.7) \quad m : V \rightarrow \mathfrak{g}^* : v \mapsto [\xi \mapsto \frac{1}{2}\omega(\xi v, v)].$$

One can show that V is a MFSR if and only if “almost” all G -invariants on V are pull-backs of coadjoint invariants, i.e., $\mathcal{O}(V)^G$ is a finitely generated $\mathcal{O}(\mathfrak{g}^*)^G$ -module.

The central result about multiplicity free symplectic representation is the following theorem proved in [Kn]:

1.1. Theorem. *Let $G \rightarrow Sp(V)$ be multiplicity free symplectic representation. Then V is cofree, i.e., the algebra of invariants $\mathcal{O}(V)^G$ is a polynomial ring and $\mathcal{O}(V)$ is a free $\mathcal{O}(V)^G$ -module.*

Observe that this has an immediate corollary, namely that also the algebra $\mathcal{W}(V)^G$ is a polynomial ring and that $\mathcal{W}(V)$ is a free $\mathcal{W}(V)^G$ -module (left or right).

The identification of $\mathcal{O}(V)^G$ with a polynomial ring can be made more precise. Recall that $\mathcal{O}(\mathfrak{g}^*)^G$ can be identified with $\mathcal{O}(\mathfrak{t}^*)^W$ where $\mathfrak{t} \subseteq \mathfrak{g}$ is a Cartan subalgebra and W is the Weyl group. Then one can find a subspace $\mathfrak{a}^* \subseteq \mathfrak{t}^*$ and a subgroup $W_V \subseteq N_W(\mathfrak{a}^*)/C_W(\mathfrak{a}^*)$ with $\mathcal{O}(V)^G \cong \mathcal{O}(\mathfrak{a}^*)^{W_V}$ and such that

$$(1.8) \quad \begin{array}{ccc} V & \xrightarrow{m} & \mathfrak{g}^* \\ \downarrow & & \downarrow \\ \mathfrak{a}^*/W_V & \rightarrow & \mathfrak{t}^*/W \end{array}$$

commutes. Moreover, W_V is a reflection group.

2. Statement of the classification

Before we state the result of our classification we have to develop some terminology. For ease of notation we state all of our results in terms of the reductive Lie algebra \mathfrak{g} . The n -dimensional commutative Lie algebra will be denoted by \mathfrak{t}^n . We will only consider algebraic representations, i.e., those which come from a representation of the corresponding group G . This is only an issue if \mathfrak{g} is not semisimple.

First, we focus on symplectic representations of a fixed Lie algebra \mathfrak{g} .

Definition: a) A symplectic representation is called *indecomposable* if it is not isomorphic to the sum of two non-trivial symplectic representations.

b) Let V be a symplectic representation. Then V is said to be of *type 1* if V is irreducible as a \mathfrak{g} -module. It is of *type 2* if $V = U \oplus U^*$ where U is an irreducible \mathfrak{g} -module not admitting a symplectic structure and the form is (1.5).

¹ Actually, m is the composition of $\text{gr } \mathcal{U}(\tilde{\rho})$ with the isomorphism $V \rightarrow V^*$ induced by ω .

2.1. Theorem.

- a) Every indecomposable symplectic representation is either of type 1 or 2.
- b) Assume two symplectic representation are isomorphic as \mathfrak{g} -modules. Then they are isomorphic as symplectic representations.
- c) Every symplectic representation is a direct sum of finitely many indecomposable symplectic representations. The summands are unique up to permutation.

Proof: Let V be a symplectic representation and $U \subseteq V$ an irreducible submodule. If $\omega|_U \neq 0$ then $V = U \oplus U^\perp$. In particular, V has a type 1 summand. If $\omega|_U = 0$ then U is in the kernel of $V \rightarrow U^* : v \mapsto \omega(v, \cdot)$. By complete reducibility, there is an inclusion $U^* \hookrightarrow V$ such that ω restricted to $\bar{U} := U \oplus U^*$ is the standard form. Any symplectic structure ω_0 on U induces an isomorphism $U \rightarrow U^* : u \mapsto u^*$. Moreover,

$$(2.1) \quad \omega(u_1 + u_1^*, u_2 + u_2^*) = \langle u_1^*, u_2 \rangle - \langle u_2^*, u_1 \rangle = \omega_0(u_1, u_2) - \omega_0(u_2, u_1) = 2\omega_0(u_1, u_2).$$

This means that ω restricted to the diagonal sitting in $U \oplus U \cong \bar{U}$ is non-zero bringing us back to the first case. Therefore, we may assume that U does not admit a symplectic structure. Since $V = \bar{U} \oplus \bar{U}^\perp$ we see that V has a type 2 summand. This already proves part a).

Let $(\tilde{V}, \tilde{\omega})$ be a second symplectic representation and assume $V \cong \tilde{V}$ as \mathfrak{g} -module. Then \tilde{V} has a summand \tilde{U} isomorphic to U . Assume first that U is symplectic. Then we can choose \tilde{U} such that $\tilde{\omega}|_{\tilde{U}} \neq 0$. Since the symplectic structure of an irreducible module is unique up to a scalar we can find an symplectic isomorphism $U \rightarrow \tilde{U}$. The orthogonal spaces $U^\perp \cong V/U$ and $\tilde{U}^\perp = \tilde{V}/\tilde{U}$ are isomorphic as \mathfrak{g} -modules. By induction, they are isomorphic as symplectic representations. Thus we get a symplectic isomorphism

$$(2.2) \quad V = U \oplus U^\perp \xrightarrow{\sim} \tilde{U} \oplus \tilde{U}^\perp = \tilde{V}.$$

The same argument works if U does not have a symplectic structure; just replace U by \bar{U} in (2.2). This completes the proof of b). The preceding discussion also shows how to read off the components of a symplectic representation from its decomposition as \mathfrak{g} -module. This proves c). □

Because of part b) we won't need to explicitly specify the symplectic structure in our tables.

Now we set up notation for when \mathfrak{g} varies.

Definition: Let $\rho_1 : \mathfrak{g}_1 \rightarrow \mathfrak{sp}(V_1)$, $\rho_2 : \mathfrak{g}_2 \rightarrow \mathfrak{sp}(V_2)$ be two symplectic representations.

- a) V_1 and V_2 are *equivalent* if there is an symplectic isomorphism $\varphi : V_1 \rightarrow V_2$ (inducing an isomorphism $\varphi : \mathfrak{sp}(V_1) \rightarrow \mathfrak{sp}(V_2)$) such that $\rho_2(\mathfrak{g}_2) = \varphi(\rho_1(\mathfrak{g}_1))$.

- b) The *product* of V_1 and V_2 is the algebra $\mathfrak{g}_1 + \mathfrak{g}_2$ acting on $V_1 \oplus V_2$. A symplectic representation is called *connected* if it is not equivalent to the product of two non-trivial symplectic representations.

This definition of equivalence has two consequences which have to be kept in mind: first, it depends only on the image of \mathfrak{g} , i.e., the kernel is being ignored. Second, two representations which differ by an (outer) automorphism are equivalent. This holds, for example, for the two spin representations of \mathfrak{so}_{2n} .

Two symplectic representations are multiplicity free if and only if their product is. This follows, e.g., directly from the definition. Therefore, it suffices to classify connected representations. Unfortunately, a representation may be connected for a rather trivial reason. Take, e.g., finitely many (non-symplectic) representations $(\mathfrak{g}_1, U_1), \dots, (\mathfrak{g}_s, U_s)$ and form the symplectic representation of $\mathfrak{g} = \prod_i \mathfrak{g}_i$ acting on $V = \oplus_i (U_i \oplus U_i^*)$. The one-dimensional Lie algebra \mathfrak{t}^1 acts on each summand by $t \cdot (u, u^*) = (tu, -tu^*)$. This way, we get a connected representation of $\mathfrak{g} + \mathfrak{t}^1$. There are cases where $(\mathfrak{g} + \mathfrak{t}^1, V)$ is multiplicity free while (\mathfrak{g}, V) is not. Thus, we cannot simply ignore the center of \mathfrak{g} . Instead, we go up and enlarge the algebra $\mathfrak{g} + \mathfrak{t}^1$ to $\mathfrak{g} + \mathfrak{t}^s$. Then the representation becomes disconnected namely the product of the $(\mathfrak{g} + \mathfrak{t}^1, U_i \oplus U_i^*)$.

Definition: A symplectic representation $\rho : \mathfrak{g} \rightarrow \mathfrak{sp}(V)$ is *saturated* if $\rho(\mathfrak{g})$ is its own normalizer in $\mathfrak{sp}(V)$.

Every type 2 representation $U \oplus U^*$ has non-trivial endomorphisms namely \mathfrak{t}^1 acting by $t \cdot (u, u^*) = (tu, -tu^*)$. Roughly speaking, saturatedness means that $\rho(\mathfrak{g})$ contains all these endomorphisms. More precisely:

2.2. Proposition. *A symplectic representation $\rho : \mathfrak{g} \rightarrow \mathfrak{sp}(V)$ is saturated if and only if every type 1 component appears with multiplicity one and the number of type 2 components equals the dimension of the center of $\rho(\mathfrak{g})$.*

Proof: Let $V = \sum_i C_i^{n_i}$ be the decomposition of V into components such that the C_i are pairwise non-isomorphic. The centralizer of $\rho(\mathfrak{g})$ in $\mathfrak{sp}(V)$ is the product of the centralizers of the $C_i^{n_i}$. There are three cases to consider:

1. C_i is of type 1. Then the centralizer is \mathfrak{so}_{n_i} .
- 2a. $C_i = U \oplus U^*$ is of type 2 with $U \not\cong U^*$. Then the centralizer is \mathfrak{gl}_{n_i} .
- 2b. $C_i = U \oplus U^*$ is of type 2 with $U \cong U^*$. Then the centralizer is \mathfrak{sp}_{2n_i} .

Now assume that V is saturated. Then $\rho(\mathfrak{g})$ contains its centralizer. Therefore, $n_i = 1$ in all cases, there are no components of type 2b and the dimension of the center equals the number of components of type 2.

Conversely, if $n_i = 1$ for type 1 components then the dimension of the center of $\rho(\mathfrak{g})$ is at most the number of the C_i of type 2a. This implies that there are no type 2b components and all $n_i = 1$. Moreover, $\rho(\mathfrak{g})$ will contain its centralizer, i.e., is self-normalizing. \square

This proposition shows in particular that, up to equivalence, a saturated representation can be easily reconstructed from the representation of the semisimple part \mathfrak{g}' of \mathfrak{g} . More precisely, the type 1 components of (\mathfrak{g}, V) are those symplectic irreducible summands of (\mathfrak{g}', V) which occur with odd multiplicity. The rest of the irreducible summands can be paired up in the form $M \oplus M^*$. These pairs form the type 2 components of (\mathfrak{g}, V) . Moreover, $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{t}^s$ where s is the number of these pairs. For example, $(\mathfrak{g}', V) = (\mathfrak{sp}_n, (\mathbb{C}^n)^{\oplus 7})$ corresponds to the saturated representation of $\mathfrak{g} = \mathfrak{sp}_n \oplus \mathfrak{t}^3$ on $V = \mathbb{C}^n \oplus (\mathbb{C}^n \oplus \mathbb{C}^n)^{\oplus 3}$.

For that reason, we use in the sequel the following notation: let $\rho : \mathfrak{s} \rightarrow \mathfrak{gl}(U)$ be a representation of a semisimple algebra \mathfrak{s} . Then we denote the type 2 representation of $\mathfrak{g} = \mathfrak{s} + \mathfrak{t}^1$ on $U \oplus U^*$ by $T(U)$. Continuing, if U_1, U_2 are two representations of \mathfrak{s} then $T(U_1) \oplus T(U_2)$ is a representation of $\mathfrak{g} = \mathfrak{s} + \mathfrak{t}^2$. Finally, one should always keep in mind that $T(U)$ is equivalent to $T(U^*)$.

The classification of multiplicity free symplectic representations is easily reduced to the saturated case:

2.3. Theorem. *Let $\rho : \mathfrak{g} \rightarrow \mathfrak{sp}(V)$ be a multiplicity free symplectic representation. Then there is a unique reductive subalgebra $\bar{\mathfrak{g}} \subseteq \mathfrak{sp}(V)$ with $\bar{\mathfrak{g}}' \subseteq \rho(\mathfrak{g}) \subseteq \bar{\mathfrak{g}}$ such that $(\bar{\mathfrak{g}}, V)$ is saturated and multiplicity free.*

Conversely, let $(\bar{\mathfrak{g}}, V)$ be a saturated and multiplicity free symplectic representation. Let \mathfrak{c} be the center of $\bar{\mathfrak{g}}$. Then there is a unique subspace $\mathfrak{a} \subseteq \mathfrak{c}$ with the following property:

Let $\mathfrak{d} \subseteq \mathfrak{c}$ be a subspace and $\mathfrak{g} := \bar{\mathfrak{g}}' \oplus \mathfrak{d}$. Then (\mathfrak{g}, V) is multiplicity free if and only if $\mathfrak{d} + \mathfrak{a} = \mathfrak{c}$.

The proof of this and the next three theorems will be given in section 4.

Now, we are in the position to state our classification results. First the indecomposable ones:

2.4. Theorem. *All saturated indecomposable MFSRs are listed in Table 1 (type 1) and Table 2 (type 2).*

Here, and throughout the paper we are using the convenient notation that a Lie algebra denotes also its defining representation. In case of an ambiguity (e.g., \mathfrak{spin}_{2n} and E_6) one may choose either of the two candidates. For example, $\mathfrak{sp}_{2m} \otimes \mathfrak{spin}_7$ is the representation of $\mathfrak{g} = \mathfrak{sp}_{2m} + \mathfrak{so}_7$ on $V = \mathbb{C}^{2m} \otimes \mathbb{C}^8$ where \mathfrak{so}_7 acts on \mathbb{C}^8 via its spin-representation. The third fundamental representation of \mathfrak{sp}_6 is denoted by $\wedge_0^3 \mathfrak{sp}_6$.

Table 1: MFSRs of type 1

	(\mathfrak{g}, V)		rank	W_V	\mathfrak{l}	i
⟨1.1⟩	$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_p$	$2m \geq p = 2n \geq 4$	n	D_n	$\mathfrak{sp}_{2m-p} + \mathfrak{t}^n$	1+
		$2m \geq p = 2n + 1 \geq 3$	n	B_n	$\mathfrak{sp}_{2m-p} + \mathfrak{t}^n$	1+
		$2 \leq 2m < p$	m	C_m	$\mathfrak{so}_{p-2m} + \mathfrak{t}^m$	1+
⟨1.2⟩	\mathfrak{sp}_{2m}	$m \geq 1$	0	—	\mathfrak{sp}_{2m-1}	1+
⟨1.3⟩	$\mathfrak{sp}_{2m} \otimes \mathfrak{spin}_7$	$m = 1$	1	A_1	$\mathfrak{sl}_3 + \mathfrak{t}^1$	1+
		$m = 2$	3	$C_2 + A_1$	\mathfrak{t}^2	1+
		$m = 3$	6	$C_3 + B_3$	0	1+
		$m \geq 4$	7	$D_4 + B_3$	\mathfrak{sp}_{2m-8}	2+
⟨1.4⟩	$\mathfrak{sl}_2 \otimes \mathfrak{spin}_9$		2	$2A_1$	$\mathfrak{sl}_3 + \mathfrak{t}^1$	1+
⟨1.5⟩	\mathfrak{spin}_n	$n = 11$	1	A_1	\mathfrak{sl}_5	1+
		$n = 12$	1	A_1	\mathfrak{sl}_6	1+
		$n = 13$	2	B_2	$\mathfrak{sl}_3 + \mathfrak{sl}_3$	1+
⟨1.6⟩	$S^3 \mathfrak{sl}_2$		1	A_1	0	1+
⟨1.7⟩	$\wedge^3 \mathfrak{sl}_6$		1	A_1	$\mathfrak{sl}_3 + \mathfrak{sl}_3$	1+
⟨1.8⟩	$\wedge_0^3 \mathfrak{sp}_6$		1	A_1	\mathfrak{sl}_3	1+
⟨1.9⟩	$\mathfrak{sl}_2 \otimes \mathbf{G}_2$		1	A_1	$\mathfrak{sl}_2 + \mathfrak{t}^1$	1+
⟨1.10⟩	$\mathfrak{sp}_4 \otimes \mathbf{G}_2$		4	$C_2 + \mathbf{G}_2$	0	1+
⟨1.11⟩	E_7		1	A_1	E_6	1+

The tables list also the rank of V (i.e., the dimension of $\mathbb{C}[V]^{\mathfrak{g}}$) and the generic isotropy algebra \mathfrak{l} . To reduce the number of cases we define $\mathfrak{gl}_0 = \mathfrak{so}_1 = \mathfrak{sp}_0 = \mathfrak{sp}_{-1} := 0$. Moreover, \mathfrak{sp}_{2n-1} denotes the isotropy algebra of any non-zero vector in the defining representation of \mathfrak{sp}_{2n} . In the last column “ i ” we list some properties of the homomorphism $\bar{\varphi}^G : \mathcal{O}(\mathfrak{g}^*)^G \rightarrow \mathcal{O}(V)^G$, the graded version of $\varphi^G : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathcal{W}(V)^G$, which might be of interest. This homomorphism factors as

$$(2.3) \quad \mathcal{O}(\mathfrak{g}^*)^G = \mathcal{O}(\mathfrak{t}^*)^W \xrightarrow{\alpha} \mathcal{O}(\mathfrak{a}^*)^N \xrightarrow{\beta} \mathcal{O}(V)^G$$

where $\mathfrak{a}^* \subseteq \mathfrak{t}^*$ is a certain subspace (namely the span of Φ_+^t from the algorithm of section 3 below) and N is its normalizer in the Weyl group W . The group N acts on \mathfrak{a}^* always as a reflection group even though there is no a priori reason. Therefore, $\mathcal{O}(\mathfrak{a}^*)^N$ is a polynomial ring and $\mathcal{O}(V)^G$ is a free $\mathcal{O}(\mathfrak{a}^*)^N$ -module. The last column records its rank $[N : W_V]$. The map α is almost surjective in the sense that Image α and $\mathcal{O}(\mathfrak{a}^*)^N$ have the same field of fractions. The sign behind the rank signifies whether α is surjective or not. This means in

particular that $\bar{\varphi}^G$ is surjective if and only if the last column contains a 1+. It is injective, if and only if $\mathfrak{l} = 0$.

In the classification of decomposable representation it turns out that certain \mathfrak{sl}_2 -factors in \mathfrak{g} pose complications.

Table 2: MFSRs of type 2

	(\mathfrak{g}, V)		rank	W_V	\mathfrak{l}	i
$\langle 2.1 \rangle$	$T(\mathfrak{sl}_m \otimes \mathfrak{sl}_n)$	$m \geq n \geq 2$	n	A_{n-1}	$\mathfrak{gl}_{m-n} + \mathfrak{t}^{n-1}$	1+
$\langle 2.2 \rangle$	$T(\wedge^2 \mathfrak{sl}_n)$	$n = 2m \geq 4$	m	A_{m-1}	\mathfrak{sl}_2^m	1+
		$n = 2m + 1 \geq 5$	m	A_{m-1}	$\mathfrak{sl}_2^m + \mathfrak{t}^1$	1+
$\langle 2.3 \rangle$	$T(S^2 \mathfrak{sl}_n)$	$n \geq 2$	n	A_{n-1}	0	1+
$\langle 2.4 \rangle$	$T(\mathfrak{sl}_n)$	$n \geq 2$	1	—	\mathfrak{gl}_{n-1}	1+
$\langle 2.5 \rangle$	$T(\mathfrak{sp}_{2m})$	$m \geq 2$	1	—	$\mathfrak{sp}_{2m-2} + \mathfrak{t}^1$	1+
$\langle 2.6 \rangle$	$T(\mathfrak{sp}_{2m} \otimes \mathfrak{sl}_n)$	$m \geq 2, n = 2$	3	$2A_1$	$\mathfrak{sp}_{2m-4} + \mathfrak{t}^1$	1+
		$m = 2, n = 3$	5	$C_2 + A_2$	0	1+
		$m = 2, n \geq 4$	6	$C_2 + A_3$	\mathfrak{gl}_{n-4}	1+
		$m \geq 3, n = 3$	6	$A_3 + A_2$	\mathfrak{sp}_{2m-6}	2+
$\langle 2.7 \rangle$	$T(\mathfrak{so}_m)$	$m \geq 5$	2	A_1	\mathfrak{so}_{m-2}	1+
$\langle 2.8 \rangle$	$T(\mathfrak{spin}_n)$	$n = 7$	2	A_1	\mathfrak{sl}_3	1+
		$n = 9$	3	$2A_1$	\mathfrak{sl}_3	1-
		$n = 10$	2	A_1	$\mathfrak{sl}_4 + \mathfrak{t}^1$	1+
$\langle 2.9 \rangle$	$T(G_2)$		2	A_1	\mathfrak{sl}_2	1+
$\langle 2.10 \rangle$	$T(E_6)$		3	A_2	\mathfrak{so}_8	1-

Definition: A MFSR (\mathfrak{g}, V) is said to have an \mathfrak{sl}_2 -link if \mathfrak{g} has a factor which is isomorphic to \mathfrak{sl}_2 and which acts effectively on more than one component of V .

2.5. Theorem. *All connected saturated MFSRs without \mathfrak{sl}_2 -links are listed in tables 11, 12, and 22.*

In these tables we used a notation which is best explained by way of an example:

$$(2.4) \quad \mathfrak{sl}_2 \otimes \mathfrak{so}_7 \underline{\oplus} \mathfrak{spin}_7 \otimes \mathfrak{sl}_2$$

means that $\mathfrak{g} = \mathfrak{sl}_2 + \mathfrak{so}_7 + \mathfrak{sl}_2$ is acting on $V = \mathbb{C}^2 \otimes \mathbb{C}^7 \oplus \mathbb{C}^8 \otimes \mathbb{C}^2$ where the first/last \mathfrak{sl}_2 is acting on the first/last \mathbb{C}^2 only while \mathfrak{so}_7 is acting diagonally on \mathbb{C}^7 (defining representation) and \mathbb{C}^8 (spin representation). The line under the \oplus -sign means that the algebras immediately to the left and to the right are being identified and is acting diagonally.

Table 11: MFSRs of type 1–1 without \mathfrak{sl}_2 -links

	(\mathfrak{g}, V)	rank	W_V	\mathfrak{l}	i
$\langle 11.1 \rangle$	$\mathfrak{sl}_2 \otimes \mathfrak{so}_n \underline{\oplus} \mathfrak{so}_n \otimes \mathfrak{sl}_2$ $n \geq 7$	4	$2A_1 + B_2$	$\mathfrak{so}_{n-4} \subset \mathfrak{so}_n$	1+
$\langle 11.2 \rangle$	$\mathfrak{spin}_{12}^+ \underline{\oplus} \mathfrak{spin}_{12}^-$	4	$4A_1$	$\mathfrak{sl}_2 + \mathfrak{sl}_2$	2–
$\langle 11.3 \rangle$	$\mathfrak{sl}_2 \otimes \mathfrak{so}_{12} \underline{\oplus} \mathfrak{spin}_{12}$	3	$3A_1$	$\mathfrak{sl}_4 + \mathfrak{t}^1$	1–
$\langle 11.4 \rangle$	$\mathfrak{sp}_4 \otimes \mathfrak{so}_{12} \underline{\oplus} \mathfrak{spin}_{12}$	7	$C_2 + A_1 + D_4$	\mathfrak{sl}_2	1–
$\langle 11.5 \rangle$	$\mathfrak{sl}_2 \otimes \mathfrak{so}_{11} \underline{\oplus} \mathfrak{spin}_{11}$	4	$2A_1 + B_2$	$\mathfrak{sl}_3 \subset \mathfrak{so}_{11}$	1–
$\langle 11.6 \rangle$	$\mathfrak{sl}_2 \otimes \mathfrak{so}_8 \underline{\oplus} \mathfrak{spin}_8 \otimes \mathfrak{sl}_2$	3	$3A_1$	$\mathfrak{sl}_2 + \mathfrak{t}^2$	1+
$\langle 11.7 \rangle$	$\mathfrak{sp}_4 \otimes \mathfrak{so}_8 \underline{\oplus} \mathfrak{spin}_8 \otimes \mathfrak{sl}_2$	7	$C_2 + D_4 + A_1$	0	1+
$\langle 11.8 \rangle$	$\mathfrak{sl}_2 \otimes \mathfrak{so}_7 \underline{\oplus} \mathfrak{spin}_7 \otimes \mathfrak{sl}_2$	4	$2A_1 + B_2$	$\mathfrak{t}^1 \subset 0 + \mathfrak{so}_7 + \mathfrak{sl}_2$	1+
$\langle 11.9 \rangle$	$\mathfrak{sl}_2 \otimes \mathfrak{spin}_7 \underline{\oplus} \mathfrak{spin}_7 \otimes \mathfrak{sl}_2$	5	$3A_1 + B_2$	0	3+
$\langle 11.10 \rangle$	$\mathfrak{sl}_2 \otimes \mathfrak{so}_6 \underline{\oplus} \mathfrak{so}_6 \otimes \mathfrak{sl}_2$	4	$2A_1 + B_2$	$\mathfrak{t}^1 \subset \mathfrak{so}_6$	1+
$\langle 11.11 \rangle$	$\mathfrak{so}_p \otimes \mathfrak{sp}_{2m} \underline{\oplus} \mathfrak{sp}_{2m}$ $p > 2m \geq 2$	$2m$	$B_m + C_m$	\mathfrak{so}_{p-2m}	1+
	$3 \leq p = 2n \leq 2m$	p	$D_n + C_n$	\mathfrak{sp}_{2m-p-1}	1+
	$3 \leq p = 2n-1 < 2m$	p	$B_{n-1} + D_n$	\mathfrak{sp}_{2m-p-1}	1+
$\langle 11.12 \rangle$	$\wedge_0^3 \mathfrak{sp}_6 \underline{\oplus} \mathfrak{sp}_6$	2	$2A_1$	\mathfrak{sl}_2	1–
$\langle 11.13 \rangle$	$\mathfrak{spin}_7 \otimes \mathfrak{sp}_4 \underline{\oplus} \mathfrak{sp}_4$	5	$A_1 + B_2 + C_2$	0	3+
$\langle 11.14 \rangle$	$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_5 \underline{\oplus} \mathfrak{sp}_4$	$m = 1$	$2A_1$	\mathfrak{t}^1	1+
		$m \geq 2$	$2C_2$	\mathfrak{sp}_{2m-5}	1+
$\langle 11.15 \rangle$	$\mathfrak{sl}_2 \otimes \mathfrak{so}_5 \underline{\oplus} \mathfrak{so}_5 \otimes \mathfrak{sl}_2$	4	$2A_1 + B_2$	0	1+

Representations with \mathfrak{sl}_2 -links are dealt with in the next theorem.

2.6. Theorem. *All connected saturated MFSRs with \mathfrak{sl}_2 -links are obtained by taking any collection of representations from Table S and identifying any number of disjoint pairs of underlined \mathfrak{sl}_2 's. Moreover, not allowed is the identification of the two \mathfrak{sl}_2 's of (S.1) and the combination (S.9)+(S.9).*

For example, if we identify the underlined \mathfrak{sl}_2 's of (S.6) and (S.8) then we get the algebra

$$(2.5) \quad \mathfrak{g} = \mathfrak{t}^1 + \mathfrak{so}_8 + \mathfrak{sl}_2 + \mathfrak{so}_7 + \mathfrak{sl}_2$$

acting on

$$(2.6) \quad V = (\mathbb{C}^8 \oplus \mathbb{C}^8) \oplus (\mathbb{C}^8 \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^8) \oplus (\mathbb{C}^7 \otimes \mathbb{C}^2)$$

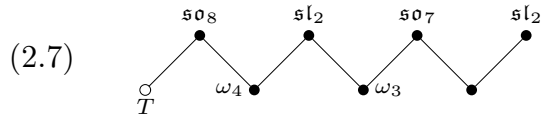
Table 12: MFSRs of type 1–2 without \mathfrak{sl}_2 -links

	(\mathfrak{g}, V)	rank	W_V	\mathfrak{l}	i
<12.1>	$\mathfrak{spin}_{12} \underline{\oplus} T(\mathfrak{so}_{12})$	4	$3A_1$	\mathfrak{sl}_4	2–
<12.2>	$\mathfrak{sl}_2 \otimes \mathfrak{so}_{10} \underline{\oplus} T(\mathfrak{spin}_{10})$	6	$2A_1 + A_3$	\mathfrak{sl}_2	1–
<12.3>	$\mathfrak{sl}_2 \otimes \mathfrak{so}_8 \underline{\oplus} T(\mathfrak{spin}_8)$	4	$3A_1$	$\mathfrak{sl}_2 + \mathfrak{t}^1$	1–
<12.4>	$\mathfrak{sl}_2 \otimes \mathfrak{so}_7 \underline{\oplus} T(\mathfrak{spin}_7)$	5	$2A_1 + B_2$	0	3+
<12.5>	$\wedge^3 \mathfrak{sl}_6 \underline{\oplus} T(\mathfrak{sl}_6 \otimes \mathfrak{sl}_2)$	6	$2A_1 + A_3$	\mathfrak{t}^1	1–
<12.6>	$\wedge^3 \mathfrak{sl}_6 \underline{\oplus} T(\mathfrak{sl}_6)$	3	$2A_1$	$\mathfrak{sl}_2 + \mathfrak{sl}_2 + \mathfrak{t}^1$	1–
<12.7>	$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_6 \underline{\oplus} T(\mathfrak{sl}_4)$	$m = 1$ 3 $m = 2$ 6 $m \geq 3$ 7	$2A_1$ $C_2 + A_3$ $2A_3$	\mathfrak{t}^2 0 \mathfrak{sp}_{2m-6}	1+ 1+ 2+
<12.8>	$\mathfrak{sl}_2 \otimes \mathfrak{so}_6 \underline{\oplus} T(\mathfrak{sl}_4 \otimes \mathfrak{sl}_2)$	6	$2A_1 + A_3$	0	1+
<12.9>	$\mathfrak{sp}_{2m} \underline{\oplus} T(\mathfrak{sp}_{2m})$	$m \geq 2$ 2	A_1	\mathfrak{sp}_{2m-3}	1+
<12.10>	$\wedge_0^3 \mathfrak{sp}_6 \underline{\oplus} T(\mathfrak{sp}_6)$	4	$A_1 + C_2$	0	3+
<12.11>	$\mathfrak{sp}_4 \underline{\oplus} T(\mathfrak{so}_5)$	3	$2A_1$	0	2+
<12.12>	$\mathfrak{sl}_2 \otimes \mathfrak{so}_5 \underline{\oplus} T(\mathfrak{sp}_4)$	4	$A_1 + B_2$	0	1+

Table 22: MFSRs of type 2–2 without \mathfrak{sl}_2 -links

	(\mathfrak{g}, V)	rank	W_V	\mathfrak{l}	i
<22.1>	$T(\mathfrak{so}_8) \underline{\oplus} T(\mathfrak{spin}_8)$	5	$3A_1$	\mathfrak{sl}_2	1–
<22.2>	$T(\wedge^2 \mathfrak{sl}_n) \underline{\oplus} T(\mathfrak{sl}_n)$	$n = 2m \geq 4$ n $n = 2m + 1 \geq 5$ n	$2A_{m-1}$ $A_m + A_{m-1}$	\mathfrak{t}^1 \mathfrak{t}^1	1– 1–
<22.3>	$T(\mathfrak{sl}_m \otimes \mathfrak{sl}_n) \underline{\oplus} T(\mathfrak{sl}_n)$	$m \geq n \geq 3$ $2n$ $2 \leq m < n$ $2m + 1$	$2A_{n-1}$ $A_m + A_{m-1}$	\mathfrak{gl}_{m-n} \mathfrak{gl}_{n-m-1}	1+ 1+
<22.4>	$T(\mathfrak{sl}_n) \underline{\oplus} T(\mathfrak{sl}_n)$	$n \geq 3$ 3	A_1	\mathfrak{gl}_{n-2}	1–
<22.5>	$T(\mathfrak{sp}_{2m}) \underline{\oplus} T(\mathfrak{sp}_{2m})$	$m \geq 2$ 4	$2A_1$	\mathfrak{sp}_{2m-4}	2+

It is easier to visualize this as a graph:

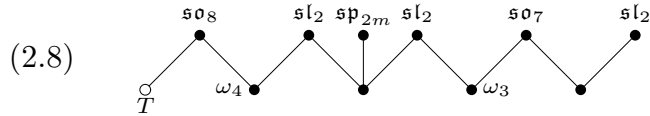


The entries (S.1) and (S.2) are special in that they possess two underlined \mathfrak{sl}_2 's. They can be used to build connected saturated MFSRs with arbitrary many components. For

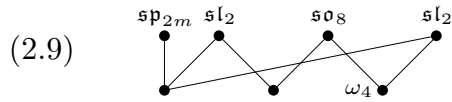
Table S

	\mathfrak{g}, X		rank	\mathfrak{l}
⟨S.1⟩	$\underline{\mathfrak{sl}}_2 \otimes \mathfrak{sp}_{2m} \otimes \underline{\mathfrak{sl}}_2$	$m = 1$	1	\mathfrak{t}^2
		$m \geq 2$	2	$\mathfrak{sp}_{2m-4} + \mathfrak{t}^2$
⟨S.2⟩	$\underline{\mathfrak{sl}}_2 \otimes \mathfrak{so}_8 \underline{\oplus} \mathfrak{spin}_8 \otimes \underline{\mathfrak{sl}}_2$		3	$\mathfrak{sl}_2 + \mathfrak{t}^2$
⟨S.3⟩	$\mathfrak{so}_n \otimes \underline{\mathfrak{sl}}_2$	$n \geq 3$	1	$\mathfrak{so}_{n-2} + \mathfrak{t}^1$
⟨S.4⟩	$\mathfrak{spin}_{12} \underline{\oplus} \mathfrak{so}_{12} \otimes \underline{\mathfrak{sl}}_2$		3	$\mathfrak{sl}_4 + \mathfrak{t}^1$
⟨S.5⟩	$\mathfrak{spin}_9 \otimes \underline{\mathfrak{sl}}_2$		2	$\mathfrak{sl}_3 + \mathfrak{t}^1$
⟨S.6⟩	$T(\mathfrak{so}_8) \underline{\oplus} \mathfrak{spin}_8 \otimes \underline{\mathfrak{sl}}_2$		4	$\mathfrak{sl}_2 + \mathfrak{t}^1$
⟨S.7⟩	$\mathfrak{spin}_7 \otimes \underline{\mathfrak{sl}}_2$		1	$\mathfrak{sl}_3 + \mathfrak{t}^1$
⟨S.8⟩	$\mathfrak{sl}_2 \otimes \mathfrak{so}_7 \underline{\oplus} \mathfrak{spin}_7 \otimes \underline{\mathfrak{sl}}_2$		4	\mathfrak{t}^1
⟨S.9⟩	$\underline{\mathfrak{sl}}_2$		0	\mathfrak{sp}_1
⟨S.10⟩	$T(\underline{\mathfrak{sl}}_2)$		1	\mathfrak{t}^1
⟨S.11⟩	$T(\mathfrak{sl}_m \otimes \underline{\mathfrak{sl}}_2)$	$m \geq 2$	2	$\mathfrak{gl}_{m-2} + \mathfrak{t}^1$
⟨S.12⟩	$T(\mathfrak{sl}_4) \underline{\oplus} \mathfrak{so}_6 \otimes \underline{\mathfrak{sl}}_2$		3	\mathfrak{t}^2
⟨S.13⟩	$\mathfrak{sp}_{2m} \otimes S^2 \underline{\mathfrak{sl}}_2$	$m \geq 1$	1	$\mathfrak{sp}_{2m-3} + \mathfrak{t}^1$
⟨S.14⟩	$T(\mathfrak{sp}_{2m} \otimes \underline{\mathfrak{sl}}_2)$	$m \geq 2$	3	$\mathfrak{sp}_{2m-4} + \mathfrak{t}^1$
⟨S.15⟩	$\mathfrak{sp}_4 \underline{\oplus} \mathfrak{so}_5 \otimes \underline{\mathfrak{sl}}_2$		2	\mathfrak{t}^1
⟨S.16⟩	$G_2 \otimes \underline{\mathfrak{sl}}_2$		1	$\mathfrak{sl}_2 + \mathfrak{t}^1$

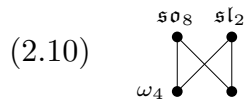
example, if one inserts (S.1) into (2.7) then one gets



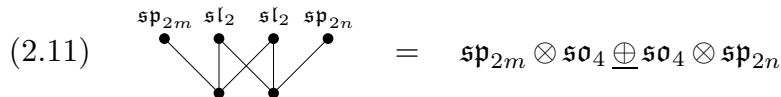
Even a circular pattern is allowed:



Finally, note the two boundary cases



and



The representation $\mathfrak{sp}_{2m} \otimes \mathfrak{so}_3 \oplus \mathfrak{so}_3 \otimes \mathfrak{sp}_{2n}$ is multiplicity free, as well, since it is the concatenation of (S.13) with itself.

3. The tool box

Before we enter the proofs of the classification theorems we collect a couple of tools. The most important one is an algorithm which allows to decide whether a given symplectic representation is multiplicity free. Moreover, it determines the rank and the generic isotropy group.

The input of the algorithm are two subsets of the weight lattice X , namely Δ , the set of roots, and Φ , the set of weights. The latter is actually a multiset, i.e., each weight is counted with its multiplicity.

Definition:

- a) $\chi \in \Phi$ is *extremal* if $\alpha \in \Delta$, $\langle \chi | \alpha^\vee \rangle > 0$ implies $\chi + \alpha \notin \Phi$.
- b) $\chi \in \Phi$ is *toroidal* if $\langle \chi | \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta$.
- c) An extremal weight χ is *singular* if $2\chi \in \Delta$ and the multiplicity of χ is one.

The algorithm: Let $\chi \in \Phi$ be an extremal weight which is neither toroidal nor singular. If no such χ exists then stop with output (Δ, Φ) . Otherwise, let $P := \{\alpha \in \Delta \mid \langle \chi | \alpha^\vee \rangle > 0\}$, $Q := \chi - P$ and perform the following replacements

$$(3.1) \quad \begin{aligned} \Delta &\rightsquigarrow \Delta \setminus (P \cup -P) \\ \Phi &\rightsquigarrow \Phi \setminus (Q \cup -Q). \end{aligned}$$

Note that if χ were toroidal then $P = \emptyset$ and nothing would happen. The extremality of χ ensures that $\Delta \setminus (P \cup -P)$ is the root system of a Levi subalgebra \mathfrak{l} . The non-singularity of χ implies $Q \cap (-Q) = \emptyset$ (in the multiset sense). This ensures that at each step, Φ is the set of weights of a symplectic \mathfrak{l} -representation.

The algorithm finishes with a pair (Δ_0, Φ_0) . Then Φ_0 is the disjoint union of the toroidal weights Φ_0^t and the singular weights Φ_0^s . Since $-\Phi_0^t = \Phi_0^t$ (and 0 has even multiplicity) we can find a subset Φ_+^t such that Φ_0^t is the disjoint union of Φ_+^t and $-\Phi_+^t$. Now, our criterion is:

3.1. Theorem. *The symplectic representation is multiplicity free if and only if the $\Phi_+^t \subseteq X$ is linearly independent. In that case, we have*

$$(3.2) \quad \dim V // G = \text{rk } V = |\Phi_+^t| = \frac{1}{2} |\Phi_0^t|.$$

Also the generic isotropy group can be determined. Let $L \subseteq G$ be the Levi subgroup with root system Δ_0 . Let $L_0 \subseteq L$ be the intersection of all $\ker \chi$ with $\chi \in \Phi_0^t$. Call two singular

weights χ_1, χ_2 equivalent if $\chi_1 - \chi_2 \in \Delta_0$. Each equivalence class determines a direct factor $\cong Sp_{2m}$ of L_0 where $2m$ is the size of the class.

3.2. Theorem. *Let $L_0 = L_1 \times Sp_{2m_1} \times \dots \times Sp_{2m_r}$ be the decomposition determined by singular weight classes. Then the generic isotropy group of G in V is (conjugate to) $L_1 \times Sp_{2m_1-1} \times \dots \times Sp_{2m_r-1}$ (where Sp_{2m-1} is an isotropy group of Sp_{2m} in $\mathbb{C}^{2m} \setminus \{0\}$).*

Proofs: Let $\chi \in \Phi$ be an non-singular extremal weight. Then χ is dominant with respect to a system $\Delta^+ \subseteq \Delta$ of positive roots. Let $\Delta_+, \Delta_0, \Delta_- \subseteq \Delta$ be the set of roots α with $\langle \chi | \alpha^\vee \rangle > 0, = 0, < 0$, and let $\mathfrak{p}_u, \mathfrak{l}, \mathfrak{p}_u^- \subseteq \mathfrak{g}$ be the corresponding subalgebras. Let $v_0 \in V$ be a weight vector for χ and choose a weight vector $v_0^- \in V$ whose weight is $-\chi$ with $\omega(v_0, v_0^-) = 1$. Then $V_0 := (\mathfrak{p}_u^- v_0)^\perp \cap (\mathfrak{p}_u v_0^-)^\perp$ is a symplectic \mathfrak{l} -representation. Let Φ_0 be the set of weights of V_0 . Moreover, we have $\Delta_0 = \Delta \setminus (P \cup -P)$ and $\Phi_0 = \Phi \setminus (Q \cup -Q)$. Thus, the next step in the algorithm encodes the representation (\mathfrak{l}, V_0) .

It has been shown in [Kn] that (\mathfrak{g}, V) is multiplicity free if and only if (\mathfrak{l}, V_0) is. Moreover, the generic isotropy groups are the same. Therefore, both properties can be read off of the output (Δ_0, Φ_0) of the algorithm which corresponds to a symplectic representation of a Levi subgroup \mathfrak{l} on a space V_0 on which every weight is either toroidal or singular. If χ is a singular dominant weight then 2χ is a root. This means that χ belongs to the defining representation of a subalgebra of type C_m . Therefore, we can decompose $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{sp}_{2m_1} + \dots + \mathfrak{sp}_{2m_r}$ and $V_0 = V_1 \oplus \mathbb{C}^{2m_1} \oplus \dots \oplus \mathbb{C}^{2m_r}$ where V_1 is a sum of one-dimension \mathfrak{l}_1 -modules. This implies $V_1 = U \oplus U^*$ where \mathfrak{l}_1 acts on U with the characters Φ_+^t . Thus (\mathfrak{l}, V_0) is multiplicity free if and only if (\mathfrak{l}_1, U) is a multiplicity free action if and only if Φ_+^t is linearly independent. The claim about the generic isotropy group is also obvious from this description. \square

Examples: 1. Let $G = \mathfrak{sl}_6 + \mathfrak{t}^2$ and $V = \wedge^3 U \oplus (U \oplus U^*) \oplus (U \oplus U^*)$ where $U = \mathbb{C}^6$ is the defining representation of \mathfrak{sl}_6 and \mathfrak{t}^2 acts according to the pattern $(0, s, -s, t, -t)$. Then

$$(3.3) \quad \begin{aligned} \Delta &= \{\varepsilon_i - \varepsilon_j\} \\ \Phi &= \{\varepsilon_i + \varepsilon_j + \varepsilon_k\} \cup \{\pm(\varepsilon_i + \eta)\} \cup \{\pm(\varepsilon_i + \eta')\} \end{aligned}$$

where $i, j, k = 1, \dots, 6$ are pairwise distinct. Moreover, the relation $\sum_i \varepsilon_i = 0$ holds. We start with the extremal weight $\chi = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Then

$$(3.4) \quad P = \{\varepsilon_i - \varepsilon_j \mid i = 1, 2, 3; j = 4, 5, 6\}$$

and we are left with

$$(3.5) \quad \begin{aligned} \Delta &= \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq 3\} \cup \{\varepsilon_i - \varepsilon_j \mid 4 \leq i \neq j \leq 6\} \\ \Phi &= \{\pm(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)\} \cup \{\pm(\varepsilon_i + \eta)\} \cup \{\pm(\varepsilon_i + \eta')\} \end{aligned}$$

Now put successively $\chi = \varepsilon_1 + \eta$, $\chi = \varepsilon_4 + \eta$, $\chi = \varepsilon_2 + \eta'$, and $\chi = \varepsilon_5 + \eta'$. The final result is

$$(3.6) \quad \begin{aligned} \Delta &= \emptyset \\ \Phi &= \Phi_+^t \cup -\Phi_+^t, \Phi_+^t = \{\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_1 + \eta, \varepsilon_4 + \eta, \varepsilon_1 + \eta', \varepsilon_2 + \eta', \varepsilon_4 + \eta', \varepsilon_5 + \eta'\} \end{aligned}$$

Since Φ_+^t is not linearly independent, the representation is not multiplicity free.

2. The other example is $G = \mathfrak{sp}_{2m} + \mathfrak{sp}_4$ acting on $V = \mathbb{C}^{2m} \otimes \mathbb{C}^5 \oplus \mathbb{C}^4$ where $\mathbb{C}^5 = \wedge_0^2 \mathbb{C}^4$ is the second fundamental representation of $\mathfrak{sp}_4(\mathbb{C})$. We assume $m \geq 2$. Then

$$(3.7) \quad \begin{aligned} \Delta &= \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i\} \cup \{\pm\varepsilon'_1 \pm \varepsilon'_2, \pm 2\varepsilon'_1, \pm 2\varepsilon'_2\} \\ \Phi &= \{\pm\varepsilon_i \pm \varepsilon'_1 \pm \varepsilon'_2, \pm\varepsilon_i\} \cup \{\pm\varepsilon'_1, \pm\varepsilon'_2\} \end{aligned}$$

We start with $\chi = \varepsilon_1 + \varepsilon'_1 + \varepsilon'_2$. Then

$$(3.8) \quad P = \{\varepsilon_1 \pm \varepsilon_j\} \cup \{2\varepsilon_1\} \cup \{\varepsilon'_1 + \varepsilon'_2, 2\varepsilon'_1, 2\varepsilon'_2\}$$

and we get

$$(3.9) \quad \begin{aligned} \Delta &= \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid i, j \geq 2\} \cup \{\pm(\varepsilon'_1 - \varepsilon'_2)\} \\ \Phi &= \{\pm(\varepsilon_1 + \varepsilon'_1 + \varepsilon'_2)\} \cup \{\pm\varepsilon_i \pm (\varepsilon'_1 - \varepsilon'_2), \pm\varepsilon_i \mid i \geq 2\} \cup \{\pm\varepsilon'_1, \pm\varepsilon'_2\} \end{aligned}$$

Since $m \geq 2$ we can repeat this process with $\chi = \varepsilon_2 + \varepsilon'_1 - \varepsilon'_2$ and get

$$(3.10) \quad \begin{aligned} \Delta &= \{\pm\varepsilon_i \pm \varepsilon_j, \pm 2\varepsilon_i \mid i, j \geq 3\} \\ \Phi &= \{\pm(\varepsilon_1 + \varepsilon'_1 + \varepsilon'_2)\} \cup \{\pm(\varepsilon_2 + \varepsilon'_1 - \varepsilon'_2)\} \cup \{\pm\varepsilon_i \mid i \geq 3\} \cup \{\pm\varepsilon'_1, \pm\varepsilon'_2\} \end{aligned}$$

Here, our algorithm terminates with $\Phi_+^t = \{\varepsilon_1 + \varepsilon'_1 + \varepsilon'_2, \varepsilon_2 + \varepsilon'_1 - \varepsilon'_2, \varepsilon'_1, \varepsilon'_2\}$ and $\Phi_0^s = \{\pm\varepsilon_i \mid i \geq 3\}$. Since Φ_+^t is linearly independent, we see that V is multiplicity free of rank 4. The Levi subgroup attached to Δ_0 is $(\mathbb{C}^*)^2 \times Sp_{2m-4}(\mathbb{C}) \times (\mathbb{C}^*)^2$. Since there is one equivalence class of singular weights we see that the generic isotropy group is $Sp_{2m-5}(\mathbb{C})$.

The algorithm is very convenient in any given special case, but it becomes quite awkward if large numbers of representations (like series) have to be handled. Therefore, we prefer to use a couple of “shortcuts” which we collect below. A very useful criterion is:

[A] *Let (\mathfrak{g}, V) be a MFSR. Then $\dim V \leq \dim \mathfrak{g} + \text{rk } \mathfrak{g}$.*

Proof: From $\dim \mathfrak{g} = |\Delta| + \text{rk } \mathfrak{g}$ we get $\dim V - \dim G + \text{rk } G = |\Phi| - |\Delta|$. Now we run the algorithm. From (3.1) we get $|\Phi| - |\Delta| = |\Phi_0| - |\Delta_0| = 2|\Phi_+^t| + |\Phi_0^s| - |\Delta_0|$. The assertion follows from $|\Phi_+^t| \leq \text{rk } \mathfrak{g}$ (since Φ_+^t is linearly independent) and $|\Phi_0^s| \leq |\Delta_0|$ (since $2\Phi_0^s \subseteq \Delta_0$). \square

The remaining criteria are dealing with the following situation: $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3$ and $V = V_1 \oplus V_2$ such that \mathfrak{g}_1 acts only on V_1 and \mathfrak{g}_3 acts only on V_2 . Let \mathfrak{l}_1 be the generic isotropy algebra of $\mathfrak{g}_1 + \mathfrak{g}_2$ in V_1 and denote its image in \mathfrak{g}_2 by \mathfrak{l}_{12} .

[B] *Assume \mathfrak{l}_1 is reductive. Then (\mathfrak{g}, V) is multiplicity free if and only if $(\mathfrak{g}_1 + \mathfrak{g}_2, V_1)$ and $(\mathfrak{l}_{12} + \mathfrak{g}_3, V_2)$ are multiplicity free.*

Proof: This is a consequence of the algorithm if we apply it first only to extremal weights of V_1 until all of them are toroidal and then to the weights of V_2 . \square

The problem with criterion **[B]** is that $(\mathfrak{l}_{12} + \mathfrak{g}_3, V_2)$ is in general not saturated. The most important special case is:

[C] *Assume $V_2 = U_2 \otimes U_3$ where U_i is an \mathfrak{g}_i -module. If $\dim U_2 \geq 3$ and $\mathfrak{l}_{12} = 0$ then V is not multiplicity free.*

Proof: This follows from **[B]** and the claim that in a MFSR no highest weight χ appears more than twice. The claim is a consequence of the algorithm since the multiplicity of χ goes down at most once and then by at most one. \square

Another useful consequence of **[B]** is

[D] *Assume*

$$\begin{aligned} & \text{either } \mathfrak{g}_1 = \mathfrak{sp}_{2m}, \mathfrak{g}_2 = \mathfrak{so}_n, V_1 = \mathbb{C}^{2m} \otimes \mathbb{C}^n, \\ & \text{or } \mathfrak{g}_1 = \mathfrak{sl}_m + \mathfrak{t}^1, \mathfrak{g}_2 = \mathfrak{sl}_n, V_1 = T(\mathbb{C}^m \otimes \mathbb{C}^n). \end{aligned}$$

Assume that V is not multiplicity free for $m = m_0 \geq 1$. Then V is not multiplicity free for all $m \geq m_0$.

Proof: Indeed $\mathfrak{l}_{12}(m) \subseteq \mathfrak{l}_{12}(m_0)$ for $m \geq m_0$ (see Tables 1 and 2). \square

The next criterion is useful in dealing with type 2 components:

[E] *Assume $\mathfrak{g}_1 = \mathfrak{sl}_2$ and $V_1 = \mathbb{C}^2 \otimes U$ where U is an \mathfrak{g}_2 -module. Suppose (\mathfrak{g}, V) is not multiplicity free. Then $(\mathfrak{t}^1 + \mathfrak{g}_2 + \mathfrak{g}_3, T(U) \oplus V_2)$ isn't either.*

Proof: Follows from $\mathfrak{t}^1 \subseteq \mathfrak{sl}_2$. \square

Sometimes, we need a more refined argument:

[F] *Same setup as in **[E]** with $\dim U \geq 2$. Let \mathfrak{l} be the generic isotropy algebra of (\mathfrak{g}, V) and let \mathfrak{h} be its image in \mathfrak{g}_1 . Then $(\mathfrak{t}^1 + \mathfrak{g}_2 + \mathfrak{g}_3, T(U) \oplus V_2)$ is multiplicity free if and only if (\mathfrak{g}, V) is multiplicity free and $\mathfrak{h} \neq 0$.*

Proof: Let (Φ_0, Δ_0) and $(\bar{\Phi}_0, \bar{\Delta}_0)$ be the outputs of the algorithm for (\mathfrak{g}, V) and $(\mathfrak{t}^1 + \mathfrak{g}_2 + \mathfrak{g}_3, T(U) \oplus V_2)$ respectively. Then $\bar{\Delta}_0 = \Delta_0$ and $\bar{\Phi}_0 = \Phi_0 \cup \{\chi - \alpha\}$ where χ is a highest weight of $\mathbb{C}^2 \otimes U$ and α is the positive root of \mathfrak{sl}_2 . Therefore, $\bar{\Phi}_+^t$ is linearly independent if and only if Φ_+^t is linearly independent, i.e. (\mathfrak{g}, V) is multiplicity free, and $\alpha \notin \langle \Phi_+^t \rangle$, i.e. $\mathfrak{h} \neq 0$. \square

4. Proofs of the classification theorems

We start with a Lemma:

4.1. Lemma. *Let $\rho : \mathfrak{g} \rightarrow \rho(V)$ be a MFSR. Then the centralizer of $\rho(\mathfrak{g})$ in $\mathfrak{sp}(V)$ is commutative.*

Proof: Recall the description of the centralizer in the proof of Proposition 2.2. Thus, we have to show: the multiplicity of a component of type 1, 2a, and 2b is ≤ 2 , ≤ 1 , and ≤ 0 , respectively. This follows from the fact that the symplectic representations $(\mathfrak{sp}(V), V^{\oplus 3})$, $(\mathfrak{gl}(U), (U \oplus U^*)^{\oplus 2})$ and $(\mathfrak{so}(U), U \oplus U)$ are not multiplicity free. \square

Proof of Theorem 2.3: The preceding lemma implies that $\bar{\mathfrak{g}}$ is necessarily the normalizer of \mathfrak{g} thereby proving its uniqueness. Now assume $(\bar{\mathfrak{g}}, V)$ is multiplicity free. Using the algorithm, we wind up with a linearly independent subset Φ_+^t of $\bar{\mathfrak{t}}^* = \mathfrak{t}^* \oplus \mathfrak{c}^*$ where \mathfrak{t} is a Cartan subalgebra of $\bar{\mathfrak{g}}'$. The condition is now that Φ_+^t stays linearly independent after restriction to $\mathfrak{t} \oplus \mathfrak{d}$. The kernel of the restriction map is $\mathfrak{d}^\perp \subseteq \mathfrak{c}^*$. Let $S := \langle \Phi_+^t \rangle \subseteq \mathfrak{t}^* \oplus \mathfrak{c}^*$. Then the condition is $S \cap \mathfrak{d}^\perp = 0$. Since $S \cap \mathfrak{d}^\perp = (S \cap \mathfrak{c}^*) \cap \mathfrak{d}^\perp$ this is equivalent to $\mathfrak{a} + \mathfrak{d} = \mathfrak{c}$ with $\mathfrak{a} = (S \cap \mathfrak{c}^*)^\perp \subseteq \mathfrak{c}$. \square

Next we state the soundness of the tables:

4.2. Lemma. *All representations in Tables 1, 2, 11, 12, and 22 are multiplicity free with the stated generic isotropy Lie algebra.*

Proof: We leave it to the reader to apply the algorithm on each item. Note that Lemma 4.7 below can be checked simultaneously without extra effort. \square

Proof of Theorem 2.4: Here we rely on previous classification work. Let V be indecomposable of type 1. Thus, by Theorem 1.1, V is an irreducible cofree representation. These have been classified by Littelmann in [Li]. If one extracts from the tables in [Li] all representations which carry a symplectic structure then one obtains exactly Table 1.

If $V = T(U)$ is of type 2 then U is a multiplicity free space (see §1). The irreducible ones have been classified by Kac [Ka] and are reproduced in Table 2. \square

During the rest of the classification we tacitly use the following observations:

- Let V_1 and V_2 be two symplectic representations. Then in order for $V = V_1 \oplus V_2$ to be multiplicity free it is necessary that both V_1 and V_2 are multiplicity free. In fact, $\mathcal{W}(V_i)^G$ is a subalgebra of $\mathcal{W}(V)^G$.
- Let (\mathfrak{g}, V) be a symplectic representation and $\mathfrak{h} \subseteq \mathfrak{g}$ a reductive subalgebra. Then in order for (\mathfrak{h}, V) to be multiplicity free it is necessary that (\mathfrak{g}, V) is multiplicity free. In fact, $\mathcal{W}(V)^\mathfrak{g}$ is a subalgebra of $\mathcal{W}(V)^\mathfrak{h}$.
- Each type 1 component appears with multiplicity one. In fact, otherwise the representation wouldn't be saturated (Proposition 2.2).
- We may assume that V contains at least one type 1 component. In fact, all representations consisting entirely of type 2 components come from multiplicity free actions (see §1). Those have been classified by Benson-Ratcliff [BR] and Leahy [Le]. Their result is reproduced in Table 22.

We continue by classifying all decomposable, connected, saturated MFSRs where exactly one simple factor of \mathfrak{g} acts on more than one component effectively. This means

$$(4.1) \quad \mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 + \dots + \mathfrak{g}_s, V = V_1 \oplus \dots \oplus V_s, \quad s \geq 2$$

where \mathfrak{g}_0 is simple and V_i is an indecomposable symplectic representation of $\mathfrak{g}_0 + \mathfrak{g}_i$ for $i = 1, \dots, s$. The main case will be $s = 2$ where we use the $\underline{\oplus}$ -notation. For $s \geq 3$ we use a notation like, e.g.,

$$(4.2) \quad \underline{\mathfrak{so}}_n \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_n \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_n \otimes \mathfrak{sl}_2$$

where \mathfrak{g}_0 is underlined. We actually show that $s = 3$ is impossible which settles also all higher cases.

4.3. Lemma. *There are no saturated MFSRs of the form (4.1) with \mathfrak{g}_0 exceptional.*

Proof: The only components involving an exceptional algebra are

$$(4.3) \quad \mathfrak{sl}_2 \otimes \mathbf{G}_2, \mathfrak{sp}_4 \otimes \mathbf{G}_2, T(\mathbf{G}_2), T(\mathbf{E}_6), \mathbf{E}_7$$

We can exclude the cases \mathbf{E}_7 , $T(\mathbf{E}_6)$, and $\mathfrak{sp}_4 \otimes \mathbf{G}_2$ [C]. The leaves

$\mathfrak{sl}_2 \otimes \mathbf{G}_2 \underline{\oplus} \mathbf{G}_2 \otimes \mathfrak{sl}_2$: not multiplicity free by [A].

$\mathfrak{sl}_2 \otimes \mathbf{G}_2 \underline{\oplus} T(\mathbf{G}_2)$: use [E]. □

4.4. Lemma. *All saturated MFSRs of the form (4.1) with $\mathfrak{g}_0 = \mathfrak{so}_n$, $n \geq 7$ are contained in Tables 11, 12, and 22.*

Proof: The only possible components not involving spin representations are

$$(4.4) \quad \mathfrak{sp}_{2m} \otimes \mathfrak{so}_n \quad (m \geq 1), \quad T(\mathfrak{so}_n)$$

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_n \underline{\oplus} \mathfrak{so}_n \otimes \mathfrak{sp}_{2p}$ ($m \geq p \geq 1$): these representations are multiplicity free for $m = p = 1$. To exclude the other cases we may assume $m = 2$ and $p = 1$ [D]. Then we use [B]. We have $\mathfrak{l}_{12} = \mathfrak{so}_{n-4} + \mathfrak{t}^2$. Thus $(\mathfrak{l}_{12} + \mathfrak{sp}_2, V_2) = \mathfrak{so}_{n-4} \otimes \mathfrak{sl}_2 \underline{\oplus} T(\mathfrak{sl}_2) \underline{\oplus} T(\mathfrak{sl}_2)$. Now we apply [C] (with V_1 equal to the last two summands).

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_n \underline{\oplus} T(\mathfrak{so}_n)$ ($m \geq 1$): not multiplicity free by [F].

The only way to combine the representations to a triple link (without spin representation) is

$\underline{\mathfrak{so}}_n \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_n \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_n \otimes \mathfrak{sl}_2$: we use [B] with V_1 being the first two summands. Then $\mathfrak{l}_{12} = \mathfrak{so}_{n-4}$. The representation $(\mathfrak{l}_{12} + \mathfrak{g}_3, V_2)$ contains $(\mathfrak{sl}_2, (\mathbb{C}^2)^{\oplus 4})$ which is not multiplicity free by [A].

Now we consider spin representations. A glance at Tables 1 and 2 shows that only the cases $n \leq 13$ have to be considered.

$n = 13$: In addition to (4.4) we have \mathfrak{spin}_{13} .

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_{13} \underline{\oplus} \mathfrak{spin}_{13}$ ($m \geq 1$): not multiplicity free. First reduce to $m = 1$ [D] and then use [A].

$T(\mathfrak{so}_{13}) \underline{\oplus} \mathfrak{spin}_{13}$: not multiplicity free [E].

$n = 12$: In addition to (4.4) we have both spin representations $\mathfrak{spin}_{12}^{\pm}$. Thus, up to isomorphism we have to check the following cases:

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_{12} \underline{\oplus} \mathfrak{spin}_{12}$ ($m \geq 1$): multiplicity free for $m = 1$ and $m = 2$. For $m \geq 3$ use [D] and [A].

$T(\mathfrak{so}_{12}) \underline{\oplus} \mathfrak{spin}_{12}$: multiplicity free.

$\mathfrak{spin}_{12}^+ \underline{\oplus} \mathfrak{spin}_{12}^-$: multiplicity free.

For triple links we have the following possibilities:

$\underline{\mathfrak{spin}}_{12} \underline{\oplus} \underline{\mathfrak{so}}_{12} \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_{12} \otimes \mathfrak{sl}_2$: we use [B] with V_1 being the last two summands. Then $\mathfrak{l}_{12} = \mathfrak{so}_8$. The restriction of \mathfrak{spin}_{12} to \mathfrak{l}_{12} is $2\mathfrak{spin}_8^+ + 2\mathfrak{spin}_8^-$, a representation which is not multiplicity free (it would be though if additionally \mathfrak{t}^2 were acting).

$\underline{\mathfrak{spin}}_{12}^+ \underline{\oplus} \underline{\mathfrak{spin}}_{12}^- \underline{\oplus} \underline{\mathfrak{so}}_{12} \otimes \mathfrak{sp}_{2m}$ ($m = 1, 2$): use [A].

$\underline{\mathfrak{spin}}_{12}^+ \underline{\oplus} \underline{\mathfrak{spin}}_{12}^- \underline{\oplus} T(\underline{\mathfrak{so}}_{12})$: use [E].

$n = 11$: In addition to (4.4) we have \mathfrak{spin}_{11} .

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_{11} \underline{\oplus} \mathfrak{spin}_{11}$ ($m \geq 1$): multiplicity free for $m = 1$. For $m \geq 2$ use [D] and [A].
 $T(\mathfrak{so}_{11}) \underline{\oplus} \mathfrak{spin}_{11}$: not multiplicity free [F].

Possible triple links:

$\underline{\mathfrak{spin}}_{11} \underline{\oplus} \underline{\mathfrak{so}}_{11} \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_{11} \otimes \mathfrak{sl}_2$: use [A].

$n = 10$: In addition to (4.4) we have $T(\mathfrak{spin}_{10})$ (observe $T(\mathfrak{spin}_{10}^+) = T(\mathfrak{spin}_{10}^-)$).

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_{10} \underline{\oplus} T(\mathfrak{spin}_{10})$ ($m \geq 1$): multiplicity free for $m = 1$. For $m \geq 2$ use [D] and [A].

Possible triple links are:

$T(\underline{\mathfrak{spin}}_{10}) \underline{\oplus} \underline{\mathfrak{so}}_{10} \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_{10} \otimes \mathfrak{sl}_2$: use [A].

$n = 9$: In addition to (4.4) we have $\mathfrak{sl}_2 \otimes \mathfrak{spin}_9$ and $T(\mathfrak{spin}_9)$.

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_9 \underline{\oplus} \mathfrak{spin}_9 \otimes \mathfrak{sl}_2$ ($m \geq 1$): not multiplicity free [D], [A].

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_9 \underline{\oplus} T(\mathfrak{spin}_9)$ ($m \geq 1$): not multiplicity free [E].

$\mathfrak{sl}_2 \otimes \mathfrak{spin}_9 \underline{\oplus} \mathfrak{spin}_9 \otimes \mathfrak{sl}_2$: not multiplicity free [A].

$\mathfrak{sl}_2 \otimes \mathfrak{spin}_9 \underline{\oplus} T(\mathfrak{spin}_9)$: not multiplicity free [E].

$\mathfrak{sl}_2 \otimes \mathfrak{spin}_9 \underline{\oplus} T(\mathfrak{so}_9)$: not multiplicity free [E].

$n = 8$: Formally, there are no new representations than (4.4) but due to triality we may replace \mathfrak{so}_8 by either spin representation. Thus, up to isomorphism we encounter the following cases:

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_8 \underline{\oplus} \mathfrak{spin}_8 \otimes \mathfrak{sp}_{2p}$ ($m \geq p \geq 1$): multiplicity free for $(m, p) = (1, 1)$ and $(2, 1)$.

Using [D] it remains to check $(m, p) = (2, 2)$ and $(3, 1)$. Both cases are handled by [A].

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_8 \underline{\oplus} T(\mathfrak{spin}_8)$: multiplicity free only for $m = 1$. Use [F] for $m \geq 2$.

For triple links we may ignore $T(\mathfrak{so}_8)$ by [E]. Thus, we are left with

$\rho_1(\underline{\mathfrak{so}}_8) \otimes \mathfrak{sp}_{2m} \underline{\oplus} \rho_2(\underline{\mathfrak{so}}_8) \otimes \mathfrak{sl}_2 \underline{\oplus} \rho_3(\underline{\mathfrak{so}}_8) \otimes \mathfrak{sl}_2$ where $m = 1, 2$ and ρ_i denotes either of the three 8-dimensional fundamental representations. In all these cases we can use [A].

$n = 7$: In addition to (4.4) we have $\mathfrak{sp}_{2m} \otimes \mathfrak{spin}_7$ ($m \geq 1$) and $T(\mathfrak{spin}_7)$. For $m \geq 3$, the image of the generic isotropy algebra of $\mathfrak{sp}_{2m} \otimes \mathfrak{spin}_7$ in \mathfrak{spin}_7 is trivial. Therefore, we may assume $m \leq 2$ [C].

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_7 \underline{\oplus} \mathfrak{spin}_7 \otimes \mathfrak{sp}_{2p}$ ($m, p \geq 1$): multiplicity free for $m = p = 1$. Using [D] and the remark above we have to check $(m, p) = (2, 1)$ and $(1, 2)$. To do this use [A].

$\mathfrak{sp}_{2m} \otimes \mathfrak{spin}_7 \underline{\oplus} \mathfrak{spin}_7 \otimes \mathfrak{sp}_{2p}$ ($m, p \geq 1$): multiplicity free for $m = p = 1$. It remains to check the three cases $(m, p) = (2, 1)$, $(1, 2)$, and $(2, 2)$ using [A].

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_7 \underline{\oplus} T(\mathfrak{spin}_7)$ ($m \geq 1$): multiplicity free only for $m = 1$. For $m \geq 2$ use [E].

$\mathfrak{sp}_{2m} \otimes \mathfrak{spin}_7 \underline{\oplus} T(\mathfrak{spin}_7)$: not multiplicity free [F].

$\mathfrak{sp}_{2m} \otimes \mathfrak{spin}_7 \underline{\oplus} T(\mathfrak{so}_7)$: not multiplicity free [F].

For triple links we may ignore $T(\mathfrak{so}_7)$ and $T(\mathfrak{spin}_7)$ by [E]. Then we are left with the representations

$\rho_1(\underline{\mathfrak{so}}_7) \otimes \mathfrak{sl}_2 \oplus \rho_2(\underline{\mathfrak{so}}_7) \otimes \mathfrak{sl}_2 \oplus \rho_3(\underline{\mathfrak{so}}_7) \otimes \mathfrak{sl}_2$ where ρ_i denotes either the defining or the spin representation. All cases are ruled out with [A]. \square

4.5. Lemma. *All saturated MFSRs of the form (4.1) with $\mathfrak{g}_0 = \mathfrak{sl}_n$, $n \geq 3$ are contained in Tables 11, 12, and 22.*

Proof: Indecomposable MFSRs of type 1 involve only \mathfrak{sl}_6 and $\mathfrak{sl}_4 = \mathfrak{so}_6$.

$n = 6$: We have to consider

$$(4.5) \quad \wedge^3 \mathfrak{sl}_6, T(\mathfrak{sl}_6 \otimes \mathfrak{sl}_m) \ (m \geq 2), T(\wedge^2 \mathfrak{sl}_6), T(S^2 \mathfrak{sl}_6), T(\mathfrak{sl}_6), T(\mathfrak{sp}_4 \otimes \mathfrak{sl}_6).$$

$\wedge^3 \mathfrak{sl}_6 \underline{\oplus} T(\mathfrak{sl}_6 \otimes \mathfrak{sl}_m)$ ($m \geq 2$): multiplicity free for $m = 2$. For $m \geq 3$ use [D] and [A].

$\wedge^3 \mathfrak{sl}_6 \underline{\oplus} T(\wedge^2 \mathfrak{sl}_6)$: use [A].

$\wedge^3 \mathfrak{sl}_6 \underline{\oplus} T(S^2 \mathfrak{sl}_6)$: use [A].

$\wedge^3 \mathfrak{sl}_6 \underline{\oplus} T(\mathfrak{sl}_6)$: multiplicity free.

$\wedge^3 \mathfrak{sl}_6 \underline{\oplus} T(\mathfrak{sl}_6 \otimes \mathfrak{sp}_4)$: use [A].

The only possible representations with triple link are:

$\wedge^3 \underline{\mathfrak{sl}}_6 \oplus T(\underline{\mathfrak{sl}}_6) \oplus T(\underline{\mathfrak{sl}}_6)$: This is a close call. The algorithm gives 7 weights which turn out to be linearly dependent (see the first example illustrating the algorithm): not multiplicity free.

$\wedge^3 \underline{\mathfrak{sl}}_6 \oplus T(\underline{\mathfrak{sl}}_6) \oplus T(\underline{\mathfrak{sl}}_6 \otimes \mathfrak{sl}_2)$: use [A].

$n = 4$: Observe $\mathfrak{sl}_4 = \mathfrak{spin}_6$ and $\wedge^2 \mathfrak{sl}_4 = \mathfrak{so}_6$. Then we get the representations

$$(4.6) \quad \mathfrak{sp}_{2m} \otimes \mathfrak{so}_6, T(\mathfrak{sl}_m \otimes \mathfrak{sl}_4) \ (m \geq 2), T(\mathfrak{so}_6), T(S^2 \mathfrak{sl}_4), T(\mathfrak{sl}_4), T(\mathfrak{sp}_4 \otimes \mathfrak{sl}_4).$$

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_6 \underline{\oplus} \mathfrak{so}_6 \otimes \mathfrak{sp}_{2p}$ ($m, p \geq 1$): same argument as for general \mathfrak{so}_n : multiplicity free only for $m = p = 1$.

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_6 \underline{\oplus} T(\mathfrak{so}_6)$ ($m \geq 1$): use [F].

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_6 \underline{\oplus} T(\mathfrak{sl}_4 \otimes \mathfrak{sl}_p)$ ($m \geq 1, p \geq 2$): multiplicity free for $(m, p) = (1, 2)$. Using [D] we can reduce the remaining cases to $(m, p) = (2, 2)$ and $(1, 3)$. Use [A] for these cases.

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_6 \underline{\oplus} T(S^2 \mathfrak{sl}_4)$: the generic isotropy group of $T(S^2 \mathfrak{sl}_4)$ is trivial. Now use [C].

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_6 \underline{\oplus} T(\mathfrak{sl}_4)$: multiplicity free for all $m \geq 1$

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_6 \underline{\oplus} T(\mathfrak{sl}_4 \otimes \mathfrak{sp}_4)$: use [D] and [A].

The possible representation with triple link are:

$\underline{\mathfrak{so}}_6 \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_6 \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_6 \otimes \mathfrak{sl}_2$: use [A].

$\underline{\mathfrak{so}}_6 \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_6 \otimes \mathfrak{sl}_2 \underline{\oplus} T(\underline{\mathfrak{sl}}_4)$: use [A].

$\underline{\mathfrak{so}}_6 \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_6 \otimes \mathfrak{sl}_2 \underline{\oplus} T(\underline{\mathfrak{sl}}_4 \otimes \mathfrak{sl}_2)$: use [A].

$\underline{\mathfrak{so}}_6 \otimes \mathfrak{sp}_{2m} \underline{\oplus} T(\underline{\mathfrak{sl}}_4) \underline{\oplus} T(\underline{\mathfrak{sl}}_4)$ ($m \geq 1$): first reduce to $m = 1$ with [D]. Then use [A].

$\underline{\mathfrak{so}}_6 \otimes \mathfrak{sl}_2 \underline{\oplus} T(\underline{\mathfrak{sl}}_4) \underline{\oplus} T(\underline{\mathfrak{sl}}_4 \otimes \mathfrak{sl}_2)$: use [A]. \square

4.6. Lemma. *All saturated MFSRs of the form (4.1) with $\mathfrak{g}_0 = \mathfrak{sp}_{2n}$, $n \geq 2$ are contained in Tables 11, 12, and 22.*

Proof: For general n , we have to consider the components

$$(4.7) \quad \mathfrak{so}_m \otimes \mathfrak{sp}_{2n} \ (m \geq 3), \ \mathfrak{sp}_{2n} \otimes \mathfrak{spin}_7, \ \mathfrak{sp}_{2n}, \ T(\mathfrak{sp}_{2n}), \ T(\mathfrak{sp}_{2n} \otimes \mathfrak{sl}_2), \ T(\mathfrak{sp}_{2n} \otimes \mathfrak{sl}_3).$$

This leaves the following possibilities with $s = 2$:

$\mathfrak{so}_m \otimes \mathfrak{sp}_{2n} \underline{\oplus} \mathfrak{sp}_{2n}$ ($m \geq 3$): multiplicity free for all $m \geq 3$, $n \geq 2$.

$\mathfrak{so}_m \otimes \mathfrak{sp}_{2n} \underline{\oplus} \mathfrak{sp}_{2n} \otimes \mathfrak{so}_p$ ($m \geq p \geq 3$): here, we directly apply the algorithm. For $m = 3$ and $p \geq 2$ we get the linearly dependent chain of weights

$$\varepsilon_1 + \varepsilon'_1, \varepsilon'_1 + \varepsilon''_1, \varepsilon'_1 - \varepsilon''_1, \varepsilon'_2 + \varepsilon''_1, \varepsilon'_2.$$

If $m \geq 4$ and $p \geq 2$ we get

$$\varepsilon_1 + \varepsilon'_1, \varepsilon_2 + \varepsilon'_2, \varepsilon'_1 + \varepsilon''_1, \varepsilon'_1 - \varepsilon''_1, \varepsilon'_2 + \varepsilon''_1, \varepsilon'_2 - \varepsilon''_1$$

which is also linearly dependent.

Observe that this argument works also for $p = 2$. Thus we get:

$\mathfrak{so}_m \otimes \mathfrak{sp}_{2n} \underline{\oplus} T(\mathfrak{sp}_{2n})$ ($m \geq 3$): not multiplicity free

$\mathfrak{so}_m \otimes \mathfrak{sp}_{2n} \underline{\oplus} \mathfrak{sp}_{2n} \otimes \mathfrak{spin}_7$ ($m \geq 3$): use $\mathfrak{spin}_7 \subset \mathfrak{so}_8$.

$\mathfrak{so}_m \otimes \mathfrak{sp}_{2n} \underline{\oplus} T(\mathfrak{sp}_{2n} \otimes \mathfrak{sl}_2)$ ($m \geq 3$): we use [B] with V_1 being the second summand.

Then $\mathfrak{l}_{12} = \mathfrak{sp}_{2n-4} + \mathfrak{t}^1$. This implies that $(\mathfrak{l}_{12} + \mathfrak{so}_m, V_2)$ contains four summands of \mathfrak{so}_m . Its saturation, $T(\mathfrak{so}_m) \underline{\oplus} T(\mathfrak{so}_m)$, is not multiplicity free.

$\mathfrak{so}_m \otimes \mathfrak{sp}_{2n} \underline{\oplus} T(\mathfrak{sp}_{2n} \otimes \mathfrak{sl}_3)$ ($m \geq 3$): same argument as above.

$\mathfrak{sp}_{2n} \underline{\oplus} T(\mathfrak{sp}_{2n} \otimes \mathfrak{sl}_2)$: use a refinement of the argument. This time a single \mathfrak{t}^1 is acting on \mathbb{C}^4 .

$\mathfrak{sp}_{2n} \underline{\oplus} T(\mathfrak{sp}_{2n} \otimes \mathfrak{sl}_3)$: same argument as above.

$\mathfrak{sp}_{2n} \underline{\oplus} \mathfrak{sp}_{2n} \otimes \mathfrak{spin}_7$: multiplicity free for $n = 2$. For $m \geq 3$ use [C] on the second summand.

$\mathfrak{sp}_{2n} \underline{\oplus} T(\mathfrak{sp}_{2n})$: multiplicity free for all $n \geq 1$.
 $\mathfrak{spin}_7 \otimes \mathfrak{sp}_{2n} \underline{\oplus} \mathfrak{sp}_{2n} \otimes \mathfrak{spin}_7$: use $\mathfrak{spin}_7 \subset \mathfrak{so}_8$.
 $\mathfrak{spin}_7 \otimes \mathfrak{sp}_{2n} \underline{\oplus} T(\mathfrak{sp}_{2n})$: use $\mathfrak{spin}_7 \subset \mathfrak{so}_8$.
 $\mathfrak{spin}_7 \otimes \mathfrak{sp}_{2n} \underline{\oplus} T(\mathfrak{sp}_{2n} \otimes \mathfrak{sl}_2)$: use $\mathfrak{spin}_7 \subset \mathfrak{so}_8$.
 $\mathfrak{spin}_7 \otimes \mathfrak{sp}_{2n} \underline{\oplus} T(\mathfrak{sp}_{2n} \otimes \mathfrak{sl}_3)$: use $\mathfrak{spin}_7 \subset \mathfrak{so}_8$.

The possible representation with $s = 3$ are so far:

$\underline{\mathfrak{sp}}_{2n} \oplus T(\underline{\mathfrak{sp}}_{2n}) \oplus T(\underline{\mathfrak{sp}}_{2n})$: we use **[B]** with V_1 being the last two summands. Thus $\mathfrak{l}_{12} = \mathfrak{sp}_{2n-4}$. It has a 4-dimensional fixed point space on the first summand: not multiplicity free.

$n = 6$: we have to consider additionally $\wedge_0^3 \mathfrak{sp}_6$.

$\mathfrak{so}_m \otimes \mathfrak{sp}_6 \underline{\oplus} \wedge_0^3 \mathfrak{sp}_6$: we use **[B]** with V_1 the second summand. Then $\mathfrak{l}_{12} = \mathfrak{sl}_3$. The saturation of $(\mathfrak{so}_m + \mathfrak{l}_{12}, V_2)$ is $T(\mathfrak{so}_m \otimes \mathfrak{sl}_3)$ which is not multiplicity free (since it is not in Table 2).

$\mathfrak{spin}_7 \otimes \mathfrak{sp}_6 \underline{\oplus} \wedge_0^3 \mathfrak{sp}_6$: use $\mathfrak{spin}_7 \subset \mathfrak{so}_8$.

$\mathfrak{sp}_6 \underline{\oplus} \wedge_0^3 \mathfrak{sp}_6$: multiplicity free.

$T(\mathfrak{sp}_6) \underline{\oplus} \wedge_0^3 \mathfrak{sp}_6$: multiplicity free.

$T(\mathfrak{sl}_2 \otimes \mathfrak{sp}_6) \underline{\oplus} \wedge_0^3 \mathfrak{sp}_6$: use **[A]**.

$T(\mathfrak{sl}_3 \otimes \mathfrak{sp}_6) \underline{\oplus} \wedge_0^3 \mathfrak{sp}_6$: use **[A]**.

The possible representations with triple link are:

$\wedge_0^3 \underline{\mathfrak{sp}}_6 \oplus \underline{\mathfrak{sp}}_6 \oplus T(\underline{\mathfrak{sp}}_6)$: use **[A]**.

$\wedge_0^3 \underline{\mathfrak{sp}}_6 \oplus T(\underline{\mathfrak{sp}}_6) \oplus T(\underline{\mathfrak{sp}}_6)$: use **[A]**.

$n = 4$: Observe $\mathfrak{sp}_4 = \mathfrak{spin}_5$. Then we get additionally $\mathfrak{sp}_{2m} \otimes \mathfrak{so}_5$, $\mathfrak{sp}_4 \otimes \mathfrak{G}_2$, $T(\mathfrak{sp}_4 \otimes \mathfrak{sl}_m)$ ($m \geq 2$), $T(\mathfrak{so}_5)$

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_5 \underline{\oplus} \mathfrak{sp}_4 \otimes \mathfrak{so}_p$ ($m \geq 1, p \geq 3$): Using **[D]** we may assume $m = 1$. We use **[B]** with V_1 being the first summand. Then $\mathfrak{l}_{12} = \mathfrak{so}_3 + \mathfrak{t}^1$. The restriction of V_2 to $\mathfrak{l}_{12} + \mathfrak{so}_p$ is $T(\mathfrak{sl}_2 \otimes \mathfrak{so}_p)$ which is not multiplicity free.

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_5 \underline{\oplus} \mathfrak{sp}_4 \otimes \mathfrak{spin}_7$ ($m \geq 1$): use $\mathfrak{spin}_7 \subset \mathfrak{so}_8$.

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_5 \underline{\oplus} \mathfrak{sp}_4$ ($m \geq 1$): multiplicity free.

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_5 \underline{\oplus} T(\mathfrak{sp}_4)$ ($m \geq 1$): multiplicity free for $m = 1$. For the other cases use **[D]** and then **[A]**.

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_5 \underline{\oplus} \mathfrak{so}_5 \otimes \mathfrak{sp}_{2p}$ ($m \geq p \geq 1$): multiplicity free for $m = p = 1$. The argument for general \mathfrak{so}_n works here as well.

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_5 \underline{\oplus} T(\mathfrak{sp}_4 \otimes \mathfrak{sl}_p)$ ($m \geq 1, p \geq 2$): first, we reduce to $m = 1$ **[D]**. Then, as in the first case, we obtain $\mathfrak{sl}_2 + \mathfrak{sl}_p + \mathfrak{t}^2$ acting on $T(\mathbb{C}^2 \otimes \mathbb{C}^p) \oplus T(\mathbb{C}^2 \otimes \mathbb{C}^p)$ which is not multiplicity free.

$\mathfrak{sp}_{2m} \otimes \mathfrak{so}_5 \underline{\oplus} T(\mathfrak{so}_5)$ ($m \geq 1$): not multiplicity free by [F].

$\mathfrak{so}_m \otimes \mathfrak{sp}_4 \underline{\oplus} T(\mathfrak{sp}_4 \otimes \mathfrak{sl}_p)$ ($m \geq 3, p \geq 2$): for $p \geq 3$, the image of $\mathfrak{l}(V_2)$ in \mathfrak{sp}_4 is 0. The case $p = 2$ has already been dealt with.

$\mathfrak{spin}_7 \otimes \mathfrak{sp}_4 \underline{\oplus} T(\mathfrak{sp}_4 \otimes \mathfrak{sl}_p)$ ($p \geq 2$): use $\mathfrak{spin}_7 \subset \mathfrak{so}_8$.

$\mathfrak{sp}_4 \underline{\oplus} T(\mathfrak{sp}_4 \otimes \mathfrak{sl}_p)$: for $p \geq 3$ argue as above. For $p = 2$ use [A].

$\mathfrak{so}_m \otimes \mathfrak{sp}_4 \underline{\oplus} T(\mathfrak{so}_5)$ ($m \geq 3$): use [E].

$\mathfrak{spin}_7 \otimes \mathfrak{sp}_4 \underline{\oplus} T(\mathfrak{so}_5)$: use $\mathfrak{spin}_7 \subset \mathfrak{so}_8$

$\mathfrak{sp}_4 \underline{\oplus} T(\mathfrak{so}_5)$: multiplicity free.

The possible representation with triple link are:

$$\underline{\mathfrak{so}}_5 \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_5 \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_5 \otimes \mathfrak{sl}_2,$$

$$\underline{\mathfrak{so}}_5 \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_5 \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{sp}}_4,$$

$$\underline{\mathfrak{so}}_5 \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{so}}_5 \otimes \mathfrak{sl}_2 \underline{\oplus} T(\underline{\mathfrak{sp}}_4),$$

$$\underline{\mathfrak{so}}_5 \otimes \mathfrak{sl}_2 \underline{\oplus} \underline{\mathfrak{sp}}_4 \underline{\oplus} T(\underline{\mathfrak{sp}}_4),$$

$$\underline{\mathfrak{so}}_5 \otimes \mathfrak{sl}_2 \underline{\oplus} T(\underline{\mathfrak{sp}}_4) \underline{\oplus} T(\underline{\mathfrak{sp}}_4), \text{ and}$$

$$\underline{\mathfrak{sp}}_4 \underline{\oplus} T(\underline{\mathfrak{sp}}_4) \underline{\oplus} T(\underline{\mathfrak{sp}}_4): \text{ all of them can be handled with [A].} \quad \square$$

Proof of Theorem 2.5: First, assume that V has only two components. If more than one simple factor $\not\cong \mathfrak{sl}_2$ acts effectively on both components then it must be obtained by identifying simple factors of \mathfrak{g}_1 with simple factors of \mathfrak{g}_2 . Going through Tables 11, 12, and 22 one sees that this is impossible without creating an \mathfrak{sl}_2 -link. Thus, these tables contain in fact all two-component, connected, saturated MFSRs.

If V has three components then we have already seen that no simple factor of \mathfrak{g} can act effectively on all of them. The other scenario is

$$(4.8) \quad \mathfrak{g} = \mathfrak{g}'_0 + \mathfrak{g}''_0 + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3, V = V_1 \oplus V_2 \oplus V_3$$

where \mathfrak{g}'_0 and \mathfrak{g}''_0 are simple, V_1 is an $\mathfrak{g}_1 + \mathfrak{g}'_0$ -module, V_2 is an $\mathfrak{g}'_0 + \mathfrak{g}_2 + \mathfrak{g}''_0$ -module and V_3 is an $\mathfrak{g}''_0 + \mathfrak{g}_3$ -module. I claim that no multiplicity free representations of that form exist.

Both representations $(\mathfrak{g}'_0 + \mathfrak{g}''_0 + \mathfrak{g}_1 + \mathfrak{g}_2, V_1 \oplus V_2)$ and $(\mathfrak{g}'_0 + \mathfrak{g}''_0 + \mathfrak{g}_2 + \mathfrak{g}_3, V_2 \oplus V_3)$ have to occur in one of the Tables 11, 12, or 22. Here are the only possibilities to combine two of them:

$$\mathfrak{spin}_{12} \underline{\oplus} \mathfrak{so}_{12} \otimes \mathfrak{sp}_4 \underline{\oplus} \mathfrak{sp}_4,$$

$$\mathfrak{sl}_2 \otimes \mathfrak{spin}_8 \underline{\oplus} \mathfrak{so}_8 \otimes \mathfrak{sp}_4 \underline{\oplus} \mathfrak{sp}_4,$$

$$T(\mathfrak{sl}_4) \underline{\oplus} \mathfrak{so}_6 \otimes \mathfrak{sp}_{2m} \underline{\oplus} \mathfrak{sp}_{2m} \quad (m \geq 2), \text{ and}$$

$$\mathfrak{sp}_4 \underline{\oplus} \mathfrak{so}_5 \otimes \mathfrak{sp}_{2m} \underline{\oplus} \mathfrak{sp}_{2m} \quad (m \geq 2): \text{ in all cases use [C] on the first two summands.}$$

Thus, all possibilities are exhausted and Theorem 2.5 is proven. \square

Before we start with the proof of Theorem 2.6, we state the significance of Table S.

4.7. Lemma. *Let (\mathfrak{g}, V) be a saturated MFSR without \mathfrak{sl}_2 -link and Ω a non-empty subset of the set of \mathfrak{sl}_2 -factors of \mathfrak{g} acting non-trivially on V . Let Φ_+^t be “the” output of the algorithm applied to (\mathfrak{g}, V) and let A be the set of simple roots of the \mathfrak{sl}_2 -factors in Ω . Assume $\Phi_+^t \cup A$ is linearly independent and that Ω is maximal with that property. Then (V, Ω) is equivalent to an entry of Table S where we indicate membership in Ω by underlining.*

Proof: This is easily checked using the algorithm. Observe that the condition to be checked is independent of the choices during the algorithm since it can be rephrased as: let $\tilde{\mathfrak{g}} \subset \mathfrak{g}$ be the subalgebra obtained by replacing each \mathfrak{sl}_2 -factor in Ω by its Cartan subalgebra. Then $(\tilde{\mathfrak{g}}, V)$ is still multiplicity free (see the proof of criterion [F]). \square

Proof of Theorem 2.6: Let (\mathfrak{g}, V) be a saturated symplectic representation and let $\mathfrak{s} \subseteq \mathfrak{g}$ be an \mathfrak{sl}_2 -link. Let $V = V_1 \oplus \dots \oplus V_r$ be the decomposition of V into components. Assume \mathfrak{s} acts non-trivially precisely on V_1, \dots, V_s with $2 \leq s \leq r$. Let $\bar{\mathfrak{g}}$ equal \mathfrak{g} but with \mathfrak{s} replaced by $\mathfrak{s}_1 + \dots + \mathfrak{s}_s$ where each $\mathfrak{s}_i \cong \mathfrak{sl}_2$. Then $(\bar{\mathfrak{g}}, V)$ is still a saturated symplectic representation and none of the \mathfrak{s}_i is an \mathfrak{sl}_2 -link. Conversely, \mathfrak{g} is obtained from $\bar{\mathfrak{g}}$ by replacing $\mathfrak{s}_1 + \dots + \mathfrak{s}_s \subseteq \bar{\mathfrak{g}}$ by its diagonal. Observe that, by construction, each \mathfrak{s}_i acts on a different component of V (this explains the first exception of the theorem). We want to find a criterion for when (\mathfrak{g}, V) is multiplicity free. Clearly, $(\bar{\mathfrak{g}}, V)$ has to be multiplicity free to begin with.

For $i = 1, \dots, s$ let α_i be the simple positive root of \mathfrak{s}_i and χ_i be a highest weight in V_i . First of all, at most one component V_i can be of type (S.9) since otherwise (\mathfrak{g}, V) wouldn't be saturated (this explains the second exception of the theorem). Thus, if V_1 is not of type (S.9) then χ_1 is an extremal weight which is neither toroidal nor singular. Now we apply our algorithm first to χ_1 and then χ_2, \dots, χ_s . If we do that with respect to the root system of $\bar{\mathfrak{g}}$ then we have to delete, among others, $\chi_1 - \alpha_1, \dots, \chi_s - \alpha_s$. If we do it with respect to the root system of \mathfrak{g} then we keep $\chi_2 - \alpha_2, \dots, \chi_s - \alpha_s$ and we have to identify $\alpha_1 = \dots = \alpha_s$. Since χ_1, \dots, χ_s are being kept in any case we see that s can't be bigger than 2. This shows that the identifications have to be pairwise and disjoint.

Thus, we have $s = 2$. Then Φ_+^t (the output with respect to \mathfrak{g}) differs from $\bar{\Phi}_+^t$ (the output with respect to $\bar{\mathfrak{g}}$) by the additional weight $\chi_2 - \alpha_2$ and the identification $\alpha_1 = \alpha_2$. This implies that (\mathfrak{g}, V) is multiplicity free if and only if $\bar{\Phi}_+^t \cup \{\alpha_1, \alpha_2\}$ is linearly independent.

Since $s = 2$, a general saturated MFSR is obtained from a collection of representation from tables 1 through 22 by pairwise identifying various \mathfrak{sl}_2 's. Let $(\tilde{\mathfrak{g}}, \tilde{V})$ one of these items and let $\mathfrak{s}_1, \dots, \mathfrak{s}_s$ be those \mathfrak{sl}_2 -factors of $\tilde{\mathfrak{g}}$ which are being identified with other \mathfrak{sl}_2 -factors.

Then an identification pattern is permissible if and only the union of Φ_+^t with the roots of the \mathfrak{s}_i is linearly independent. This is precisely the condition for being member of Table S (Lemma 4.7). \square

5. References

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