Integrality of Two Variable Kostka Functions

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1. Introduction

Macdonald defined in [M1] a remarkable class of symmetric polynomials $J_\lambda(x;q,t)$ which depend on two parameters and interpolate between many families of classical symmetric polynomials. For example $Q_\lambda(x;t) = J_\lambda(x;0,t)$ are the Hall-Littlewood polynomials which themselves specialize for $t = 0$ to Schur functions s_λ. Also Jack polynomials arise by taking $q = t^\alpha$ and letting t tend to 1.

The Hall-Littlewood polynomials are orthogonal with respect to a certain scalar product $\langle\cdot,\cdot\rangle_t$. The scalar products $K_{\lambda\mu} = \langle s_\lambda, h_\mu\rangle_0$ are known as Kostka numbers. Since it has an interpretation as a weight multiplicity of a GL_n-representation, it is a natural number. Also the scalar products $K_{\lambda\mu}(t) = \langle s_\lambda, Q_\mu(x;t)\rangle_t$ are important polynomials in t. They satisfy $K_{\lambda\mu}(1) = K_{\lambda\mu}$ and their coefficients have been proven to be natural numbers by Lascoux-Schützenberger [LS].

Macdonald conjectured in [M1] that even the scalar products

$$K_{\lambda\mu}(q,t) = \langle s_\lambda, J_\mu(x;q,t)\rangle_t$$

are polynomials in q and t with non-negative integers as coefficients. In this note we prove that $K_{\lambda\mu}(q,t)$ is at least a polynomial. The positivity seems to be much deeper and is, to my knowledge, unsolved up to know.

Our main tool has also been introduced by Macdonald. Following a construction of Opdam [O1] in the Jack polynomial case, he constructed a family E_λ of non-symmetric polynomials. Here the indexing set is now all of $\mathbb{N}^n$. They have properties which are very similar to those of the symmetric functions, but in a sense they are easier to work with. In particular, we exhibit a very simple recursion formula (a “creation operator”) in terms of Hecke operators for them. This formula enables us to prove an analogous conjecture for

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these non-symmetric polynomials, at least what polynomiality concerns. At the end, we obtain our main result by symmetrization.

The proof follows very closely a proof of an analogous conjecture for Jack polynomials. In this case we were even able to settle positivity. This will appear along with a new combinatorial formula for Jack polynomials as a joint paper with S. Sahi. The main difference to the Jack polynomial case is, of course, the appearence of Hecke operators. Furthermore, we introduce a certain basis which is the non-symmetric analogue of Hall-Littlewood polynomials. But the main point is again a very special creation operator which in the present case is

$$\Phi f(z_1, \ldots, z_n) = z_n f(q^{-1} z_n, z_1, \ldots, z_{n-1}).$$

Meanwhile, this appoach has been partially generalized by Cherednik [C2] to arbitrary root systems.

Finally, let me mention that independendly there appeared other proofs of the integrality of q, t-Kostka coefficients by Garsia-Tesler [GT], Kirillov-Noumi [KN], and Sahi [S1].

2. Definitions

Let $\Lambda := \mathbb{N}^n$ and Λ^+ be the subset of all partitions. For $\lambda = (\lambda_i) \in \Lambda$ we put $|\lambda| := \sum_i \lambda_i$ and $l(\lambda) := \max\{i \mid \lambda_i \neq 0\}$ (with $l(0) := 0$). To each $\lambda \in \Lambda$ corresponds a monomial z^λ.

There is a (partial) order relation on Λ. First, recall the usual order on the set Λ^+: we say $\lambda \geq \mu$ if $|\lambda| = |\mu|$ and

$$\lambda_1 + \lambda_2 + \ldots + \lambda_i \geq \mu_1 + \mu_2 + \ldots + \mu_i \quad \text{for all } i = 1, \ldots, n.$$

This order relation is extended to all of Λ as follows. Clearly, the symmetric group $W = S_n$ on n letters acts on Λ and for every $\lambda \in \Lambda$ there is a unique partition λ^+ in the orbit $W\lambda$. For all permutations $w \in W$ with $\lambda = w\lambda^+$ there is a unique one, denoted by w_λ, of minimal length. We define $\lambda \geq \mu$ if either $\lambda^+ > \mu^+$ or $\lambda^+ = \mu^+$ and $w_\lambda \leq w_\mu$ in the Bruhat order of W. In particular, λ^+ is the unique *maximum* of $W\lambda$.

Let k be the field $\mathbb{C}(q, t)$, where q and t are formal variables and let $\mathcal{P} := k[z_1, \ldots, z_n]$ be the polynomial ring and $\mathcal{P}' = k[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$ the Laurent polynomial ring over k. There are involutory automorphisms $\iota : x \mapsto \bar{x}$ of $k/\mathbb{C}$ with $\bar{q} = q^{-1}$, $\bar{t} := t^{-1}$, and a ι-semilinear involution $\mathcal{P}' \mapsto \mathcal{P}' : f \mapsto \overline{f}$ with $\overline{z_i} = z_i^{-1}$. For $f \in \mathcal{P}$ let $[f]_1$ be its constant term. Fix an $r \in \mathbb{N}$ and put $t = q^r$. Then Cherednik (see [C1], [M3] §5) defines a certain Laurent polynomial $\delta_r(x; t)$ and a pairing on $\mathcal{P}'$ which is Hermitian with respect to ι by

$$\langle f, g \rangle := [\delta_r f \bar{g}]_1.$$

Non-symmetric Macdonald polynomials are defined by the following theorem.

2.1. Theorem. ([M3] §6) *For every $\lambda \in \Lambda$ there is a unique polynomial $E_\lambda(z;q,t) \in \mathcal{P}$ satisfying*

i) $E_\lambda(z) = z^\lambda + \sum_{\mu\in\Lambda:\mu<\lambda} c_{\lambda\mu}(q,t) z^\mu$ *and*

ii) $\langle E_\lambda(z), z^\mu\rangle_r = 0$ *for all $\mu \in \Lambda$ with $\mu < \lambda$ and almost all $r \in \mathbb{N}$.*

Moreover, the collection $\{E_\lambda \mid \lambda \in \Lambda\}$ forms a k-linear basis of $\mathcal{P}$.

The symmetric group W acts also on $\mathcal{P}$ in the obvious way. We are going to define a basis of $\mathcal{P}^W$, the algebra of symmetric functions, which is parametrized by Λ^+. One basis is already given by the monomial symmetric functions m_λ, $\lambda \in \Lambda^+$. Next, we define the symmetric Macdonald polynomials:

2.2. Theorem. ([M3] 1.5) *For every $\lambda \in \Lambda^+$ there is a unique symmetric polynomial $P_\lambda(z;q,t) \in \mathcal{P}^W$ satisfying*

i) $P_\lambda(z) = m_\lambda + \sum_{\mu\in\Lambda^+:\mu<\lambda} c'_{\lambda\mu}(q,t) m_\mu$ *and*

ii) $\langle P_\lambda(z), m_\mu\rangle_r = 0$ *for all $\mu \in \Lambda^+$ with $\mu < \lambda$ and almost all $r \in \mathbb{N}$.*

Moreover, the collection $\{P_\lambda \mid \lambda \in \Lambda\}$ forms an k-linear basis of $\mathcal{P}^W$.

3. The Hecke algebra

The scalar product above is *not* symmetric in the variables z_i. Therefore, we define operators which replace the usual W-action. Let $s_i \in W$ be the i-th simple reflection. First, we define the operators $N_i := (z_i - z_{i+1})^{-1}(1 - s_i)$ and then

$$H_i := s_i - (1-t)N_i z_i \qquad \overline{H}_i := s_i - (1-t) z_{i+1} N_i.$$

They satisfy the relations

$$H_i - \overline{H}_i = t - 1; \qquad H_i \overline{H}_i = t.$$

This means that both H_i and $-\overline{H}_i$ solve the equation $(x+1)(x-t) = 0$. Also the braid relations hold

$$\begin{aligned} H_i H_{i+1} H_i &= H_{i+1} H_i H_{i+1} \qquad && i = 1, \ldots, n-2 \\ H_i H_j &= H_j H_i && |i-j| > 1 \end{aligned}$$

This means that the algebra generated by the H_i is a Hecke algebra of type A_{n-1}. For all this see [M3]. For compatibility, let me remark that our parameter t is t^2 in [M3] and our H_i is tT_i there. Furthermore, the simple roots (or rather their exponentials) are z_{i+1}/z_i.

The connection with the W-action is that we get the same set of invariants in the following sense: $f \in \mathcal{P}^W$ if and only if $\overline{H}_i(f) = f$ for all i if and only if $H_i(f) = t\,f$ for all i.

The braid relations imply that $H_w := H_{i_1} \dots H_{i_k}$ is well defined if $w = s_{i_1} \dots s_{i_k} \in W$ is a reduced decomposition and similarly for $\overline{H}_w$. The following relations hold:

$$z_{i+1} H_i = \overline{H}_i z_i; \quad H_i z_{i+1} = z_i \overline{H}_i \qquad i = 1, \dots, n-1.$$

Now, we introduce the operator Δ by

$$\Delta f(z_1, \dots, z_n) := f(q^{-1} z_n, z_1, \dots, z_{n-1}).$$

The following relations are easily checked

$$\begin{aligned} \Delta z_{i+1} &= z_i \Delta \qquad i = 1, \dots, n-1; \\ \Delta z_1 &= q^{-1} z_n \Delta; \\ \Delta H_{i+1} &= H_i \Delta \qquad i = 1, \dots, n-2; \\ \Delta^2 H_1 &= H_{n-1} \Delta^2. \end{aligned}$$

The last relation means that if we define $H_0 := \Delta H_1 \Delta^{-1} = \Delta^{-1} H_{n-1} \Delta$ then $H_0, \dots, H_{n-1}$ generate an affine Hecke algebra while $H_1, \dots, H_{n-1}, \Delta$ generate the extended affine Hecke algebra corresponding to the weight lattice of GL_n. In particular, there must be a family of n commuting elements: the Cherednik operators. In our particular case, they (or rather their inverses) have a very nice explicit form. For $i = 1, \dots, n$ put

$$\xi_i^{-1} := \overline{H}_i \overline{H}_{i+1} \dots \overline{H}_{n-1} \Delta H_1 H_2 \dots H_{i-1}.$$

We have the following commutation relations

$$\begin{aligned} &\xi_{i+1} H_i = \overline{H}_i \xi_i; \quad H_i \xi_{i+1} = \xi_i \overline{H}_i \qquad i = 1, \dots, n-1; \\ &\xi_i H_j = H_j \xi_i \qquad i \neq j, j+1. \end{aligned}$$

The relation to Macdonald polynomials is as follows. For $\lambda \in \Lambda$ define $\bar{\lambda} \in k^n$ as $\bar{\lambda}_i := q^{\lambda_i} t^{-k_i}$ where

$$k_i := \#\{j = 1, \dots, i-1 \mid \lambda_j \geq \lambda_i\} + \#\{j = i+1, \dots, n \mid \lambda_j > \lambda_i\}.$$

The following Lemma is easy to check:

3.1. Lemma. ([M3] 4.13,5.3) *a) The action of ξ_i on $\mathcal{P}$ is triangular. More precisely, $\xi_i(z^\lambda) = \bar{\lambda}_i z^\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} z^\mu$.*

b) Let $r \in \mathbb{N}$. Then, with respect to the scalar product $\langle \cdot, \cdot \rangle_r$, the adjoints of H_i, Δ, ξ_i are H_i^{-1}, Δ^{-1}, ξ_i^{-1} respectively.

Since $\bar{\lambda} = \bar{\mu}$ implies $\lambda = \mu$, we immediately get:

3.2. Corollary. *The ξ_i admit a simultaneous eigenbasis E_λ of the form $z^\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} z^\mu$ with eigenvalue $\bar{\lambda}_i$. Moreover, these functions coincide with those defined in Theorem 2.1.*

Actually, this gives a proof of Theorem 2.1 and that the ξ_i commute pairwise.

4. Recursion relations

Observe that the operators z_i and ξ_i behave very similarly. The only exception is, that there is no simple commutation rule for $\Delta\xi_1$ as there is for Δz_1. Therefore, we introduce another operator which simply is $\Phi := z_n\Delta = q\Delta z_1$. Then one easily checks the relations

$$\Phi\xi_{i+1} = \xi_i\Phi; \quad \Phi\xi_1 = q\xi_n\Phi.$$

This implies:

4.1. Theorem. *Let $\lambda \in \Lambda$ with $\lambda_n \neq 0$ and put $\lambda^* := (\lambda_n - 1, \lambda_1, \ldots, \lambda_{n-1})$. Then $E_\lambda = q^{\lambda_n - 1}\Phi(E_{\lambda^*})$.*

Observe that also the following relations hold:

$$\begin{aligned} \Phi z_{i+1} &= z_i\Phi \qquad i = 1, \ldots, n-1; \\ \Phi H_{i+1} &= H_i\Phi \qquad i = 1, \ldots, n-2; \\ \Phi^2 H_1 &= H_{n-1}\Phi^2. \end{aligned}$$

This means that if $\tilde{H}_0 := \Phi H_1 \Phi^{-1} = \Phi^{-1} H_{n-1}\Phi$ then $\tilde{H}_0, H_1, \ldots, H_{n-1}$ generate another copy of the affine Hecke algebra, but note $\tilde{H}_0 \neq H_0$! Indeed, H_0 is acting on $\mathcal{P}$, while $\tilde{H}_0$ acts only on $\mathcal{P}'$.

The theorem above works as a recursion relation for E_λ if $\lambda_n \neq 0$. The next (well known) lemma tells how to permute two entries of λ.

4.2. Theorem. *Let $\lambda \in \Lambda$ and s_i a simple reflection.*
a) Assume $\lambda_i = \lambda_{i+1}$. Then $H_i(E_\lambda) = tE_\lambda$ and $\overline{H}_i(E_\lambda) = E_\lambda$.
b) Let $\lambda_i > \lambda_{i+1}$ and $x := 1 - \bar{\lambda}_i/\bar{\lambda}_{i+1}$. Then $xE_\lambda = [xH_i + 1 - t]E_{s_i(\lambda)}$.

Proof: a) Let $\mu < \lambda$. Then it follows from properties of the Bruhat order that $\overline{H}_i(z^\mu)$ is a linear combination of z^ν with $\nu < \lambda$. Hence, Lemma 3.1a implies that $H_i(E_\lambda)$ is orthogonal to all z^μ with $\mu < \lambda$. By definition it must be proportional to E_λ. The assertion follows by comparing the coeffients of z^λ.

b) Denote the right hand side by E. Since the coefficients of z^λ are the same, it suffices to prove that E is an eigenvector of ξ_j with eigenvalue $\bar{\lambda}_j$. This is only non-trivial for $j = i, i+1$. Let $j = i$. Then

$$\begin{aligned}\xi_i(E) &= \xi_i[x(\overline{H}_i + t - 1) + (1-t)]E_{s_i(\lambda)} \\ &= [x\xi_i\overline{H}_i + (1-t)\bar{\lambda}_i/\bar{\lambda}_{i+1}\xi_i]E_{s_i(\lambda)} = [x\bar{\lambda}_i H_i + (1-t)\bar{\lambda}_i]E_{s_i(\lambda)} = \bar{\lambda}_i E.\end{aligned}$$

The case $j = i+1$ is handled similarly. □

These two formulas suffice to generate all E_λ but we will need a more refined version.

4.3. Lemma. *Let $\lambda \in \Lambda$ with $\lambda_a \neq 0$, $\lambda_{a+1} = \ldots = \lambda_b = 0$. Let $\lambda^\#$ equal λ except that the a-th and b-th components are interchanged. Then*

$$(1 - \bar{\lambda}_a t^a)E_\lambda = [\overline{H}_a\overline{H}_{a+1}\ldots\overline{H}_{b-1} - \bar{\lambda}_a t^a H_a H_{a+1}\ldots H_{b-1}]E_{\lambda^\#}.$$

Proof: We prove this by induction on $b-a$. Let λ' equal λ but with the a-th and $b-1$-st components interchanged. Then $\bar{\lambda}'_{b-1} = \bar{\lambda}_a$ and $\bar{\lambda}'_b = t^{-b+1}$. With $x' := 1 - \bar{\lambda}_a t^{b-1}$, Theorem 4.2b implies $x'E_{\lambda'} = [x'H_{b-1} + 1 - t]E_{\lambda^\#}$. Hence, with $x := 1 - \bar{\lambda}_a t^a$, we get by induction

$$\begin{aligned}xx'E_\lambda &= [\overline{H}_a\ldots\overline{H}_{b-2} - (1-x)H_a\ldots H_{b-2}][x'H_{b-1} + 1 - t]E_{\lambda^\#} \\ &= [x'\overline{H}_a\ldots\overline{H}_{b-2}(\overline{H}_{b-1} + t - 1) + (1-t)\overline{H}_a\ldots\overline{H}_{b-2} - \\ &\quad - x'(1-x)H_a\ldots H_{b-1} - (1-t)(1-x)H_a\ldots H_{b-2}]E_{\lambda^\#}\end{aligned}$$

Now observe $H_i(E_{\lambda^\#}) = tE_{\lambda^\#}$ for $i = a, \ldots, b-2$ and $\overline{H}_i(E_{\lambda^\#}) = E_{\lambda^\#}$ by Theorem 4.2a. Hence the expression above becomes

$$\begin{aligned}&[x'\overline{H}_a\ldots\overline{H}_{b-1} - x'(1-x)H_a\ldots H_{b-1} + \\ &\qquad + (t-1)x' + (1-t) - (1-t)(1-x)t^{b-a-1}]E_{\lambda^\#}.\end{aligned}$$

The lemma follows since the constant terms cancel out. □

For $m = 1, \ldots, n$ define the operators

$$\begin{aligned}A_m &:= H_m H_{m+1}\ldots H_{n-1}\Phi \\ \overline{A}_m &:= \overline{H}_m\overline{H}_{m+1}\ldots\overline{H}_{n-1}\Phi\end{aligned}$$

Then we obtain:

4.4. Corollary. *Let $\lambda \in \Lambda$ with $m := l(\lambda) > 0$. Put $\lambda^* := (\lambda_m - 1, \lambda_1, \ldots, \lambda_{m-1}, 0, \ldots, 0)$. Then $(1 - \bar{\lambda}_m t^m)E_\lambda = q^{\lambda_m - 1}[\overline{A}_m - \bar{\lambda}_m t^m A_m]E_{\lambda^*}$.*

5. Integrality

To remove the denominators in the coefficients of E_λ we use a normalization as follows. Recall, that the *diagram* of $\lambda \in \Lambda$ is the set of points (usually called *boxes*) $s = (i, j) \in \mathbb{Z}^2$ such that $1 \le i \le n$ and $1 \le j \le \lambda_i$. For each box s we define the *arm-length* $a(s)$ and *leg-length* $l(s)$ as

$$\begin{aligned} a(s) &:= \lambda_i - j \\ l'(s) &:= \#\{k = 1, \ldots, i-1 \mid j \le \lambda_k + 1 \le \lambda_i\} \\ l''(s) &:= \#\{k = i+1, \ldots, n \mid j \le \lambda_k \le \lambda_i\} \\ l(s) &:= l'(s) + l''(s) \end{aligned}$$

If $\lambda \in \Lambda^+$ is a partition then $l'(s) = 0$ and $l''(s) = l(s)$ is just the usual leg-length. Now we define

$$\mathcal{E}_\lambda := \prod_{s \in \lambda} (1 - q^{a(s)+1} t^{l(s)+1}) E_\lambda.$$

With this normalization, we obtain:

5.1. Theorem. *With the notation of Corollary 4.4 let* $X_\lambda := q^{\lambda_m - 1}(\overline{A}_m - \bar{\lambda}_m t^m A_m)$*. Then* $\mathcal{E}_\lambda = X_\lambda(\mathcal{E}_{\lambda^*})$*.*

Proof: It suffices to check the coefficient of z^λ. The factor $q^{\lambda_m - 1}$ cancels the effect of $\Phi = z_n \Delta$ on this coefficient. The diagram of λ^* is obtained from λ by taking the last non-empty row, removing the first box s_0 and putting the rest on top. It is easy to check that arm-length and leg-length of the boxes $s \neq s_0$ don't change. Hence the assertion follows from Corollary 4.4 since the factor corresponding to s_0 is just $1 - \bar{\lambda}_m t^m$. □

Now we can state our first integrality result:

5.2. Corollary. *Let* $\mathcal{E}_\lambda = \sum_\mu c_{\lambda\mu} z^\mu$*. Then* $c_{\lambda\mu} \in \mathbb{Z}[t, q]$*.*

Proof: Every $\mathcal{E}_\lambda$ is obtained by repeated application of operators X_μ. Looking at the definition of Δ we conclude that the $c_{\lambda\mu}$ are in $\mathbb{Z}[q, q^{-1}, t]$. We exclude the possibility of negative powers of q. For this write

$$\Phi = z_n \Delta = z_n \overline{H}_{n-1}^{-1} \ldots \overline{H}_1^{-1} \xi_1^{-1}$$

Now, $\mathcal{E}_{\lambda^*}$ is an eigenvector of ξ_1^{-1} with eigenvalue $(\bar{\lambda}_m^*)^{-1} = q^{-\lambda_m + 1} t^a$ with some $a \in \mathbb{Z}$. This shows (by induction) that $q^{\lambda_m - 1}\Phi(\mathcal{E}_{\lambda^*})$ doesn't contain negative powers of q. □

Our goal is a more refined integrality result. For this we replace the monomial basis z^λ by a more suitable one. For $\lambda\in\Lambda$ let $w_\lambda, \tilde w_\lambda$ be the shortest permutations such that $\lambda^+ := w_\lambda^{-1}(\lambda)$ is dominant (i.e. a partition) and $\lambda^- := \tilde w_\lambda^{-1}(\lambda)$ is antidominant. Now we define the *t-monomial* $\mathfrak{m}_\lambda := \overline{H}_{\tilde w^\lambda}(z^{\lambda^-})$. The reason for this is that the action of the H_i becomes nicer:

$$H_i(\mathfrak{m}_\lambda) = \begin{cases} t\mathfrak{m}_{s_i(\lambda)} & \text{if } \lambda_i \ge \lambda_{i+1} \\ \mathfrak{m}_{s_i(\lambda)} + (t-1)\mathfrak{m}_\lambda & \text{if } \lambda_i < \lambda_{i+1} \end{cases}$$

$$\overline{H}_i(\mathfrak{m}_\lambda) = \begin{cases} t\mathfrak{m}_{s_i(\lambda)} + (1-t)\mathfrak{m}_\lambda & \text{if } \lambda_i > \lambda_{i+1} \\ \mathfrak{m}_{s_i(\lambda)} & \text{if } \lambda_i \le \lambda_{i+1} \end{cases}$$

This is easily proved by induction on the length of $\tilde w_\lambda$. Moreover, it is easy to see that the transition matrix between t-monomials and ordinary monomials is unitriangular.

Now we define a length function on Λ by $L(\lambda) := l(w_\lambda) = \#\{(i,j) \mid i<j, \lambda_i < \lambda_j\}$.

5.3. Lemma. *The function* $\mathfrak{m}_\lambda^{(0)} = \sum_\mu t^{L(\mu)}\mathfrak{m}_\mu$*, where the sum runs through all permutations* μ *of* λ*, is symmetric.*

Proof: It suffices to prove $H_i(\mathfrak{m}_\lambda^{(0)}) = t\mathfrak{m}_\lambda^{(0)}$ for all i. This follows easily from the explicit description of the action given above. □

Clearly, the symmetric t-monomials $\mathfrak{m}_\lambda^{(0)}$, $\lambda\in\Lambda^+$ (later we will see that they are nothing else than the Hall-Littlewood polynomials) also have a unitriangular transition matrix to the monomial symmetric functions m_λ.

For technical reasons we need also partially symmetric t-monomials. Let $0\le m\le n$ fixed and $\lambda\in\Lambda$ let $\lambda' := (\lambda_1,\ldots,\lambda_m)$ and $\lambda'' := (\lambda_{m+1},\ldots,\lambda_n)$. We also write $\lambda = \lambda'\lambda''$. Let $\Lambda^{(m)} \subseteq \Lambda$ be the set of those λ such that λ'' is a partition. For these elements we form

$$\mathfrak{m}_\lambda^{(m)} := \sum_\mu t^{L(\mu)}\mathfrak{m}_{\lambda'\mu}$$

where μ runs through all permutations of λ''.

For $k\in\mathbb{N}$ let $\varphi_k(t) := (1-t)(1-t^2)\ldots(1-t^k)$. Then $[k]! := \varphi_k(t)/(1-t)^k$ is the t-factorial. For a partition μ we define $m_i(\mu) := \#\{j \mid \mu_j = i\}$ and $b_\mu(t) := \prod_{i\ge1}\varphi_{m_i(\lambda)}(t)$. Now we define the augmented partially symmetric t-monomial as $\tilde{\mathfrak{m}}_\lambda^{(m)} := b_{\lambda''}(t)\mathfrak{m}_\lambda^{(m)}$. The key result of this paper is

5.4. Theorem. *For* $m \ge l(\lambda)$ *consider the expansion* $\mathcal{E}_\lambda = \sum\limits_{\mu\in\Lambda^{(m)}} c_{\lambda\mu}\tilde{\mathfrak{m}}_\mu^{(m)}$*. Then the coefficients* $c_{\lambda\mu}$ *are in* $\mathbb{Z}[q,t]$*.*

Proof: By Corollary 5.2, the only denominators which can occur are products of factors of the form $1-t^k$ (or divisors thereof). In particular, it suffices to show that the $c_{\lambda\mu}$ are in $\mathbb{Z}[q,q^{-1},t,t^{-1}]$. Therefore, as in the proof of Corollary 5.2, we may replace Φ by

$$\Phi' := z_n\overline{H}_{n-1}^{-1}\ldots\overline{H}_1^{-1} = t^{1-n}z_nH_{n-1}\ldots H_1$$

and A_m, $\overline{A}_m$ by

$$\begin{aligned}A'_m &:= H_mH_{m+1}\ldots H_{n-1}\Phi'\\ \overline{A}'_m &:= \overline{H}_m\overline{H}_{m+1}\ldots\overline{H}_{n-1}\Phi'\end{aligned}$$

Therefore, the theorem is proved with the next lemma. □

5.5. Lemma. *a) Every $\tilde{\mathfrak{m}}_\lambda^{(m)}$ is a linear combination of $\tilde{\mathfrak{m}}_\mu^{(m+1)}$ with coefficients in $\mathbb{Z}[t]$.*
b) The operators A'_m and $\overline{A}'_m$ commute with $H_{m+1},\ldots,H_{n-1}$. In particular, they leave the space stable which is spanned by all $\tilde{\mathfrak{m}}_\lambda^{(m)}$.
c) The matrix coefficients of A'_m and $\overline{A}'_m$ with respect to this basis are in $\mathbb{Z}[t,t^{-1}]$.

Proof: a) is obvious and b) an easy consequence of commutation and braid relations. For c) let us start with another lemma.

5.6. Lemma. *Let $\lambda\in\Lambda$ with $\lambda_n\neq 0$. Then $\Phi'(\mathfrak{m}_{\lambda^*}) = t^{-a}\mathfrak{m}_\lambda$ where $a=\#\{i<n\mid \lambda_i>\lambda_n\}$.*

Proof: Using braid and commutation relations, one verifies $\Phi'\overline{H}_i = \overline{H}_{i-1}\Phi'$ for all $i>1$. Now assume $\lambda_{i-1}>\lambda_i$ for some $i<n$. Then using induction on $l(\tilde{w}_\lambda)$ we may assume the result is correct for $\mathfrak{m}_{s_{i-1}(\lambda)}$. Thus,

$$\begin{aligned}\Phi'(\mathfrak{m}_{\lambda^*}) =&\Phi'\overline{H}_i(\mathfrak{m}_{s_i(\lambda^*)}) = \overline{H}_{i-1}\Phi'(\mathfrak{m}_{s_i(\lambda^*)}) =\\ =&t^{-a}\overline{H}_{i-1}\mathfrak{m}_{s_{i-1}(\lambda)} = t^{-a}\mathfrak{m}_\lambda\end{aligned}$$

Thus, we may assume $\lambda_1\le\ldots\le\lambda_{n-1}$. Let $l:=\lambda_n-1$ and $b:=\max\{i<n\mid\lambda_i\le l\} = n-1-a$. For simplicity, we denote $\mathfrak{m}_\lambda$ by $[\lambda]$. Thus

$$\begin{aligned}&t^{n-1}\Phi'[\lambda^*] = z_nH_{n-1}\ldots H_1[\lambda^*] = t^b\overline{H}_{n-1}\ldots\overline{H}_{b+1}z_{b+1}[\ldots,\lambda_b,l,\lambda_{b+1},\ldots] =\\ &= t^b\overline{H}_{n-1}\ldots\overline{H}_{b+1}[\ldots,\lambda_b,l+1,\lambda_{b+1},\ldots] = t^b[\ldots,\lambda_{n-1},l+1]\end{aligned}$$

□

We are continuing with the proof of Lemma 5.5. By part a), $\overline{A}'_m\mathfrak{m}_\lambda^{(m)}$ is symmetric in the variables $z_{m+1},\ldots,z_n$. Therefore, it suffices to investigate the coefficient of $[\mu]$ where μ'' is an anti-partition. Let $[\nu]$ be a typical term of $\mathfrak{m}_\lambda^{(m)}$, i.e., $\nu'=\lambda'$ and ν'' is a permutation of λ''. Let $l := \lambda_1 + 1 = \nu_1 + 1$. Looking how Φ' and $\overline{H}_i$ act, we see that $\overline{A}'_m[\nu]$ is a linear combination of terms

$$[\mu] = [s_{i_1} \ldots s_{i_r}(\nu_2, \ldots, \nu_n, l)]$$

where $m \le i_1 < \ldots < i_r < n$. If $r = n - m$ then $b_{\mu''} = b_{\nu''} = b_{\lambda''}$ and we are done.

Otherwise there is $m \le j < n$ maximal such that $j - 1 \notin \{i_1, \ldots, i_r\}$. Since $s_j \ldots s_{n-1}(\nu_2, \ldots, \nu_n, l) = (\ldots, \nu_j, l, \nu_{j+1}, \ldots)$ we necessarily have $\nu_j > l$. We are only interested in the case where μ'' is an anti-partition. So ν_j has to be moved all the way to the m-th position. This means $(i_1, \ldots, i_r) = (m, m+1, \ldots, j-2, j, \ldots, n-1)$. Moreover, $\nu_m \le \ldots \nu_{j-1} \le l \le \nu_{j+1} \le \nu_n$ and $\nu_j > l$. There are exactly $m_l(\nu'') + 1 = m_l(\mu'')$ such permutations ν'' of λ''. Each of them contributes $t^{-a}(1-t)t^{L(\nu'')}$ for the coefficient of $[\mu]$ in $\overline{A}'_m[\lambda]$. With j, the length $L(\nu'')$ runs through a consecutive segment $b, \ldots, b + m_l(\nu'')$ of integers. So $[\mu]$ gets the factor $t^{b-a}(1-t)(1+t+\ldots+t^{m_l(\nu'')}) = t^{b-a}(1-t^{m_l(\mu'')})$. This shows that the coefficient of $[\mu]$ in $b_{\lambda''}A_m\mathfrak{m}_\lambda^{(m)}$ is divisible by $b_{\mu''}$.

The case for A_m is completely analogous and the details are left to the reader. The only change is that one only looks for the coefficient of $[\mu]$ where μ'' is a partition. □

The proof gives actually a little bit more:

5.7. Corollary. *For all $\lambda \in \Lambda$ we have $\mathcal{E}_\lambda(z; 0, t) = \mathfrak{m}_\lambda$.*

Proof: Assume $m = l(\lambda)$ and look at $\overline{A}'_m\mathfrak{m}_{\lambda^*}$. In the notation of the proof above, the second case (where some s_j is missing) can not occur since then $\nu_j = 0$ would be greater than $l > 0$. So we get $\mathfrak{m}_\lambda = \overline{A}'_m\mathfrak{m}_{\lambda^*}$. This means that $\mathfrak{m}_\lambda$ satisfies the same recursion relation as $\mathcal{E}_\lambda$ with $q = 0$. □

6. The symmetric case and Kostka numbers

Finally we come to the integrality properties of the symmetric polynomial P_λ where $\lambda \in \Lambda^+$. For this we normalize it as follows

$$J_\lambda(z; q, t) := \prod_{s \in \lambda}(1 - q^{a(s)}t^{l(s)+1})P_\lambda(z; q, t)$$

6.1. Theorem. *Let $J_\lambda(z) = \sum_\mu c_{\lambda\mu}(q, t)\tilde{\mathfrak{m}}_\mu^{(0)}$. Then the coefficients $c_{\lambda\mu}$ are in $\mathbb{Z}[q, t]$.*

Proof: Let $m := l(m)$ and consider λ^-, the anti-partition with $\lambda^-_i = \lambda_{n+1-i}$ and $\lambda^0 := (\lambda_m - 1, \ldots, \lambda_1 - 1, 0, \ldots, 0)$. Let $\mathcal{E} := q^{|\lambda|-m}\Phi^m(\mathcal{E}_{\lambda^0})$. Then $\mathcal{E}$ equals $\mathcal{E}_{\lambda^-}$, except that in the normalization factor the contributions of the first column of λ^- are missing. Put

$$J := \frac{(1-t)^m}{[m_0(\lambda)]!} \sum_{w \in W} H_w(\mathcal{E}).$$

We claim that $J = J_\lambda$. Consider the subspace V of $\mathcal{P}$ spanned by all $E_{w\lambda}$, $w \in W$. Then it follows from Lemma 3.1b and the definitions that J_λ spans V^W. This shows that J is proportional to J_λ.

To show equality we compare the coefficient of $\mathfrak{m}_\lambda$. Since λ^- is anti-dominant only those summands $H_w(\mathcal{E})$ have an $\mathfrak{m}_\lambda$-term where $w\lambda^- = \lambda$. These w form a left coset for W_λ. Therefore, summation over this coset contributes the factor

$$\sum_{w \in W_\lambda} t^{l(w)} = \prod_{i \geq 0} [m_i(\lambda)]! = \frac{[m_0(\lambda)]!}{(1-t)^m} b_\lambda(t).$$

Thus, $\mathfrak{m}_\lambda$ has in J the coefficient

$$b_\lambda(t) \prod_{\substack{s \in \lambda^- \\ s \neq (i,1)}} \left(1 - q^{a(s)+1} t^{l(s)+1}\right).$$

On the other hand, by definition, the coefficient of $\mathfrak{m}_\lambda$ in J_λ is

$$\prod_{s \in \lambda} (1 - q^{a(s)} t^{l(s)+1}).$$

Let $w = w_{\lambda^-}$, the shortest permutation which transforms λ into λ^-. This means $w(i) > w(j)$ whenever $\lambda_i > \lambda_j$ but $w(i) < w(j)$ for $\lambda_i = \lambda_j$ and $i < j$. Consider the following correspondence between boxes:

$$\lambda \ni s = (i,j) \leftrightarrow s^- = (w(i), j+1) \in \lambda^- .$$

This is defined for all s with $j < \lambda_i$. One easily verifies that $a(s) = a(s^-) + 1$ and $l(s) = l(s^-)$. This means that s and s^- contribute the same factor in the products above. What is left out of the correspondence are those boxes of λ with $j = \lambda_i$ and the first column of λ^-. The first type of these boxes contributes b_λ to the factor of J_λ. The second type doesn't contribute by construction. This shows $J_\lambda = J$.

Finally, we have to show that the coefficient of $\tilde{\mathfrak{m}}^{(0)}_\mu$ in J is in $\mathbb{Z}[q,t]$. By Corollary 5.2 it suffices to show that these coefficients are in $A := \mathbb{Z}[q, q^{-1}, t, t^{-1}]$. So we can ignore negative powers of q and t and replace $\mathcal{E}$ by $\mathcal{E}' := (\Phi')^m(\mathcal{E}_{\lambda^0})$ and similarly J by J'. Lemma 5.6 shows that $(\Phi')^m(\mathfrak{m}^{(m)}_\mu)$ equals now the symmetrization of some $\mathfrak{m}_{\mu'}$ in the *first* $n-m$ variables. Therefore, by abuse of notation let $\mu=\mu''\mu'$ where μ' are the last m components of μ and let $\mathfrak{m}_\mu^{(m)}$ be the symmetrization of $\mathfrak{m}_\mu$ in μ''.

The isotropy group of λ^- is generated by simple reflections. Hence $\mathcal{E}'$ is W_{λ^-} invariant. We may assume that μ is dominant for W_λ, i.e., $\mu_i \geq \mu_{i+1}$ whenever $\lambda_i=\lambda_{i+1}$. Let $m_{ij} := \#\{k \mid \lambda_k = i, \mu_k = j\}$. Then $\mathcal{E}'$ is an A-linear combination of

$$\frac{b_{\mu''}}{\prod_{ij}[m_{ij}]!}\sum_{w\in W_\lambda} H_w\mathfrak{m}_\mu$$

Averaging over W we obtain that J' is an A-linear combination of

$$\frac{(1-t)^m}{[m_0(\lambda)]!}\frac{b_{\mu''}}{\prod_{ij}[m_{ij}]!}\prod_i[m_i(\lambda)]!\prod_j[m_j(\mu)]!\,\mathfrak{m}_\mu^{(0)}$$

Because $b_\mu=(1-t)^{n-m_0(\mu)}\prod_{j\geq1}[m_j(\mu)]!$ and $b_{\mu''}=(1-t)^{l(\mu'')}\prod_{j\geq1}[m_{0j}]!$ the expression above equals

$$(1-t)^{m+l(\mu'')+m_0(\mu)-n}\frac{[m_0(\mu)]!}{[m_{00}]!}\prod_{i\geq1}\frac{[m_i(\lambda)]!}{\prod\limits_{j\geq0}[m_{ij}]!}\tilde{\mathfrak{m}}_\mu^{(0)}.$$

This proves the theorem. □

To put this into a more classical perspective note:

6.2. Theorem. *Let $\lambda\in\Lambda^+$. Then $\mathfrak{m}_\lambda^{(0)}(z;t)=P_\lambda(z;0,t)=P_\lambda(z;t)$ and $\tilde{\mathfrak{m}}_\lambda^{(0)}(z;t)=J_\lambda(z;0,t)=Q_\lambda(z;t)$ where $P_\lambda(z;t)$ and $Q_\lambda(z;t)$ is the Hall-Littlewood polynomial in the notation of [M2] III.*

Proof: Corollary 5.7 implies that $\mathfrak{m}_\lambda^{(0)}$ is a multiple of $P_\lambda(z;0,t)$. The leading coefficient is in both cases 1. Thus they are equal. The rest follows from $P(z;0,t)=P_\lambda(z;t)$ ([M2] VI.1) and the definitions. □

Recall that the Kostka-functions $K_{\lambda\mu}(q,t)$ form the transition matrix from the Macdonald polynomials $J_\lambda(z;q,t)$ to the t-Schur functions $S_\mu(z;t)$. It is known that the transition matrix from the $S_\mu(z;t)$ to the Hall-Littlewood polynomials $Q_\lambda(z;t)$ is unitriangular ([M2]). Hence, Theorem 6.1 can be rephrased as

6.3. Theorem. *For all $\lambda,\mu\in\Lambda^+$, we have $K_{\lambda\mu}(q,t)\in\mathbb{Z}[q,t]$.*

Remark: The duality relation between Kostka coefficients can be used to obtain the above theorem much faster, namely directly from Corollary 5.2. I would like to thank Adriano Garsia to point that out to me. But I think the proof presented above has still its virtues since it gives a very sharp integrality result for the non-symmetric polynomials

(Theorem 5.4) and shows very clearly where cancellations occur. The short argument is as follows: From Corollary 5.2 and the equality $J_\lambda = J$ in the proof of Theorem 6.1 one deduces easily that $v(t)K_{\lambda\mu}$ is in $\mathbb{Z}[q,t]$ where $v(t)$ is some polynomial with $v(0) = 1$. By [M1] VI.8.15, we have $K_{\lambda\mu}(q,t) = K_{\lambda'\mu'}(t,q)$ where λ', μ' denote the dual partitions. This implies that also $v'(q)K_{\lambda\mu} \in \mathbb{Z}[q,t]$ where $v'(q)$ is some polynomial. Now $\mathbb{Z}[q,t]$ is a UFD and the greatest common divisor of $v(t)$ and $v'(q)$ is 1 which implies Theorem 6.3.

7. Jack polynomials

Let me shortly indicate how to obtain even positivity for Jack polynomials. As already mentioned in the introduction, a detailed proof completely in the framework of Jack polynomials will appear as a joint paper with S. Sahi. There we even give a combinatorial formula in terms of certain tableaux.

Let α be an indeterminate and put formally $q = t^\alpha$. Let $p(q,t) \in \mathbb{Q}(q,t)$, $p_0 \in \mathbb{Q}$ and $k \in \mathbb{N}$. Then we write $p \xrightarrow{n,\alpha} p_0$ if $\lim\limits_{t\to 1} \frac{p(t^\alpha,t)}{(1-t)^n} = p_0$. For example, $1 - q^a t^b \xrightarrow{1,\alpha} a\alpha + b$.

Let $J_\lambda(z;\alpha)$ be a Jack polynomial. One could define it by $J_\lambda(z;q,t) \xrightarrow{|\lambda|,\alpha} J_\lambda^{(\alpha)}(z)$ ([M2] VI.10.23). There is also a non-symmetric analogue defined by $\mathcal{E}_\lambda(z;q,t) \xrightarrow{|\lambda|,\alpha} \mathcal{E}_\lambda^{(\alpha)}(z)$.

For any $\lambda \in \Lambda$ and $1 \le m \le n$ we have $\mathfrak{m}_\lambda \xrightarrow{0,\alpha} z^\lambda$ and $\mathfrak{m}_\lambda^{(m)} \xrightarrow{0,\alpha} m_\lambda^{(m)} := \sum_\mu z^{\lambda'\mu}$ where μ runs through all permutations of λ''. If $\lambda \in \Lambda^+$ let $u_\lambda := \prod_{i\ge 1} m_i(\lambda)!$. Then we have $b_\lambda \xrightarrow{l(\lambda),\alpha} u_\lambda$. In particular, $\tilde{\mathfrak{m}}_\lambda^{(m)} \xrightarrow{l(\lambda),\alpha} \tilde{m}_\lambda^{(m)} := u_{\lambda''} m_\lambda^{(m)}$. With this notation we have:

7.1. Theorem. *a) Let $\lambda \in \Lambda$ and $m = l(\lambda)$. Then there is an expansion $\mathcal{E}_\lambda^{(\alpha)} = \sum_{\mu\in\Lambda^{(m)}} c_{\lambda\mu}(\alpha)\tilde{m}_\mu^{(m)}$ with $c_{\lambda\mu}(\alpha) \in \mathbb{N}[\alpha]$ for all μ.*

b) Let $\lambda \in \Lambda^+$ and $m = l(\lambda)$. Then there is an expansion $J_\lambda^{(\alpha)} = \sum_{\mu\in\Lambda^+} c'_{\lambda\mu}(\alpha)\tilde{m}_\mu^{(0)}$ with $c'_{\lambda\mu}(\alpha) \in \mathbb{N}[\alpha]$ for all μ.

Proof: Going to the limit $t \to 1$, Theorem 5.4 and Theorem 6.1 imply $c_{\lambda\mu}, c'_{\lambda\mu} \in \mathbb{Z}[\alpha]$ (see also [M2] VI.10 Ex. 2a). It remains to show positivity. Since $J_\lambda^{(\alpha)}$ is the symmetrization of $\mathcal{E}_\lambda^{(\alpha)}$ it suffices to show positivity for the latter.

For this, we write the operator X_λ of Theorem 5.4 as $X_\lambda = q^{\lambda_m - 1}((\overline{A}_m - A_m) + (1 - \bar{\lambda}_m t^m)A_m)$. Then $X_\lambda \xrightarrow{1,\alpha} X_\lambda^1$ and we prove that all parts of X_λ^1 preserve positivity. With

$\Phi_1 := z_n s_{n-1} \dots s_1$ this follows from

$$q^{\lambda_m - 1} \xrightarrow{0,\alpha} 1,$$

$$\Phi \xrightarrow{0,\alpha} \Phi_1,$$

$$A_m \xrightarrow{0,\alpha} s_m \dots s_{n-1},$$

$$\overline{A}_m - A_m \xrightarrow{1,\alpha} \sum_{i=m}^{n-1} s_m \dots \widehat{s_i} \dots s_{n-1} \Phi_1,$$

$$1 - \bar{\lambda}_m t^m \xrightarrow{1,\alpha} \alpha\lambda_m - k + m,$$

where k is number of $i = 1, \dots, m-1$ with $\lambda_i \geq \lambda_m$. □

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